Selling Mechanisms for a Financially Constrained Buyer*

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Abstract A seller has multiple items to allocate. A buyer has private information about her values for these items and demands at most one of them; in addition, the buyer is financially constrained. A selling mechanism specifies a collection of non-linear prices, given that the buyer's types (pairs of valuation and budget) are private information. Focusing on mechanisms that never generate a deficit for the seller, we provide necessary and sufficient conditions for selling mechanisms to be prior free incentive compatible and ex post budget feasible for the buyer. These conditions inform the construction of the incentive compatible prices. We use a novel flow network approach to incentive compatibility that also takes care of budget feasibility, exploiting a subtle difference between *unrestricted incremental values* —i.e., the minimal value difference between an item assigned to the buyer by the mechanism and another alternative— and *restricted incremental values* —i.e., the minimal value difference between the assigned item and the alternative when the buyer can actually afford the alternative, given her financial position. We derive important properties on prices for any given allocation function that is implementable without deficits, and illustrate the usefulness of our approach in two settings: (i) a multi-item allocation problem with a convex type space; and (ii) a revenue maximization problem when the seller has two identical objects to allocate and the buyer's valuation may exhibit complementarities between the objects.

1 Introduction

Budget constraints are central to many economic transactions, including spectrum and natural resource auctions,[1] housing and other durable goods markets,[2] and e-commerce.[3] To cite but one example, Cramton (1995) stresses the likelihood that all firms that participated in the historic spectrum license auction, held on July 1994 by the FCC, were facing liquidity

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[1]See for instance Salant (1997) and Cramton (2010).

[2]See Ortalo-Magne and Rady (2006), more recently Han et al. (2017).

[3]See Milkman and Beshears (2009).

constraints. As important as the presence of budget constraints on the buyers' side is the fact that sellers sometimes take them explicitly into account: in Google's keyword auction and other search-engine advertising platforms, bidders are required to specify their bids as well as their daily budget limits, and Amazon's Cloud Computing service allows customers to create billing alarms to monitor their spending.[4]

Relatively few things are known about the design of auctions and selling mechanisms in the presence of budget constraints. Yet this should be seen as more than just a theoretical exercise towards more plausible modeling assumptions in economics. Indeed, classical results in the literature do not carry over to common scenarios with budget constraints. For instance, Che and Gale (1998) show the revenue dominance of the first-price auction over second-price auction in the presence of private budgets. On the positive side, some results do carry over: Dobzinski et al. (2012) show that the clinching auction due to Ausubel (2004) generalizes when bidders have publicly known budgets. Results currently available in the literature with *privately known* budgets are however restricted to special cases; e.g., single-item settings.[5] While an important baseline, auctions and other selling mechanisms for the single-item case ignore, to paraphrase Milgrom (2017), markets with complex constraints where substitutabilities and/or complementarities among goods are relevant considerations.

As an illustration, suppose a seller has two licenses to operate airfreight services between major hubs and is uncertain about what kind of buyer it encounters. Interested airfreight carriers can exploit complementarities between the licenses in different ways and have access to different funding sources. Model this as a buyer-seller situation where the seller has two items to allocate, a and a'. Item a represents allocating a single license to the buyer, whereas a' represents allocating both licenses. A small carrier with low fixed costs places a high value in obtaining one route. But because of its limited capacity, it places little marginal value in obtaining both routes instead of a single one (these values are represented as the v_1–column in the table below). At the other end, a large carrier with high fixed costs places a small value in obtaining either of the routes alone. Being able to exploit complementarities, the large carrier derives a large marginal benefit from operating both routes, instead of any single one (the values for the large carrier are represented in the v_4–column). Intermediate carriers (columns v_2 and v_3) lie somewhere in between.

	v_1	v_2	v_3	v_4
a	11	10	13	6
a'	16	20	28	26

Suppose in addition that the buyer's financial constraint is either low (5) or high (30). At what prices should licenses be sold to maximize revenue? More fundamentally, how should the seller allocate licenses? Little is known to guide the seller's decisions in situations like this one, when the buyer has privately observed budgetary restrictions and there are multiple items to sell.

We believe that the lack of general results in designing mechanisms for financially constrained buyers stems from a gap in our understanding of what, precisely, implementability

[4]On Google, see for example Edelman et al. (2007). Amazon's information comes from `http://docs.aws.amazon.com/AmazonCloudWatch/latest/DeveloperGuide/monitor_estimated_charges_with_cloudwatch.html` (retrieved January 30th, 2018).

[5]We review some of this literature at the end of this section.

means in this environment. To fill this void, this paper studies prior-free (dominant strategy) incentive compatible, deterministic selling mechanisms in a model where a seller has multiple items to allocate, and a buyer has private valuations and private budgets, which act as hard constraints. Our model, presented in Section 2, generalizes the illustration given above. A priori, we do not impose any restriction on the buyer's valuations and thus our model can be adapted to study many different situations, including multi-unit and multi-object allocation problems with or without substitutabilities and complementarities. We say that an allocation function is *implementable without deficits* if there exists a price function that is budget feasible for the buyer, generates no deficit for the seller, and makes truthful revelation incentive compatible. In other words, since the buyer's financial constraints don't bite if the seller is allowed to subsidize consumption, all mechanisms we consider are ex-post deficit-free for the seller and, at the same time, respect the buyer's ex-post budget constraints. It is easy to extend our results to *acceptable mechanisms*, i.e., selling mechanisms that are in addition ex-post individually rational.

We work with a novel flow network approach to incentive compatibility that, in addition, takes care of budget feasibility. Our conditions exploit properties of an allocation network that endows directed edges between its nodes with a *minimal* and a *maximal* capacity. To construct these capacities, we introduce the notions of *unrestricted incremental values* (linked to minimal capacities) and *restricted incremental values* (linked to maximal capacities). Roughly speaking, the unrestricted incremental value between two items, say a and a', is the minimal valuation difference between a and a' that the buyer can obtain when the selling mechanism assigns her item a. As such, it points towards the difference in prices that is sufficient to stop the buyer from deviating from the recommendation of the mechanisms and purchasing a' instead of a. This notion has appeared previously in the literature, e.g. Rochet (1987). Importantly, it includes all possible deviations to a', including those of a buyer who may not be able to afford to purchase a', because of her financial constraints. On the other hand, the restricted incremental value between a and a' is the minimal valuation difference between obtaining a and a' that the buyer gets when assigned a by the mechanism and, in addition, the buyer can actually afford to buy a'. The subtle distinction between unrestricted and restricted incremental values plays a key role in our arguments and has, to the best of our knowledge, not been exploited before.

It is important to realize that any wedge between unrestricted and restricted incremental values arises endogenously, because it responds to the provisions embedded in the allocation function. Stated differently, the distinction between incremental values, and consequently the distinction between minimal and maximal capacities, is not technical but reflects economic considerations. For instance, when there are two items to allocate, if the seller discriminates solely on ability to pay, i.e., by partitioning the buyer's type space via budgets and assigning a different item to each partition, the restricted incremental value for the low priced item is $+\infty$ (the unrestricted incremental value is always finite). The wedge between incremental values is positive and finite when the seller does not separate solely on the basis of financial constraints, instead letting a high budget type purchase the low priced item when the valuation for it exceeds a certain threshold.

In Section 3 we show that a necessary condition for implementability without deficits of an allocation function is that its corresponding allocation network admits no cycle of negative maximal capacity (Proposition 2). We also show that a sufficient condition for implementability without deficits is that the allocation network contains no cycle of negative minimal capacity (Proposition 3). We provide examples to demonstrate that our results are tight: neither is our necessary condition sufficient, nor is our sufficient condition necessary for implementability without deficits. Our proofs are elementary and do not rely on any sophisticated machinery, which allows us to obtain a clear picture of the informational constraints jointly imposed by incentive compatibility and budget feasibility. A sufficient condition is that the allocation network has no cycle of negative minimal capacity, because in this case the buyer has no profitable deviation from the alternative recommended by the mechanism, regardless of whether deviations are affordable or not. On the other hand, it is necessary for implementability that the allocation network has no cycle of negative maximal capacity, as this way of encapsulating incentive compatibility considers only affordable deviations. Extending our results to the case of acceptable mechanisms is immediate.

Although in this paper we focus on implementability, refraining from a systematic search for revenue-maximizing mechanisms, our flow network approach allows us to derive results on the structure of incentive prices in a straightforward way. For instance, as we show in Section 4, maximal capacities in the allocation network can be used to bound prices from above (Proposition 6). The intuition behind these bounds generalizes the intuition behind marginal pricing to a multi-object environment: the price for item a cannot exceed the incremental value of any other affordable alternative. Importantly, the upper bounds do not depend on ad-hoc assumptions in terms of valuations, supports or type distributions. Our techniques can therefore be used to study settings where there is correlation of valuations and budgets, or settings where goods are gross substitutes or exhibit complementarities, cases for which there are few if any results on mechanisms for financially constrained buyers. We show that if the seller has two items to allocate, the upper bounds on prices are tight (Proposition 8), and thus we can derive maximal prices associated to *any* implementable allocation function directly from maximal capacities in the allocation network.

In Section 5 we apply to our results to exploring maximal prices in two settings. In the classical house allocation problem,[6] prices are exogenously given and a mechanism searches for incentive compatible and efficient ways to allocate agents to houses (students to programs, individuals to positions, etc.). In Section 5.1 we consider the *dual* problem[7] where the allocation is fixed, responding for instance to different priority considerations (disability needs in the case of housing, merit in the case of students), and the seller searches for maximal prices that implement this assignment without deficits. Assuming a convex type domain, we find a very simple way to express maximal prices for allocation functions that satisfy our sufficiency condition for implementability without deficits. These maximal prices come directly from the allocation network.

Because our main results on implementability do not rely on the structure of the valuation space, they can be easily applied to study selling mechanisms when the seller has two units of the same good to allocate and a buyer's preferences may exhibit complementarities between them. This situation was illustrated above with the example of air freight license allocation, where the buyer has four valuations and two budgets. In Section 5.2 we derive the revenue maximizing selling mechanism for this case. We also do comparative statics on weakening the buyer's budget constraints, for instance by increasing the low budget of the buyer. We are able to make a counterintuitive observation: increasing the low budget (weakly) increases revenue from the optimal selling mechanism, but it sometimes does so by changing the buyer's

[6]See for example Hylland and Zeckhauser (1979).

[7]Our use of the duality notion is tongue-in-cheek.

assignments, for example by excluding a type with low budget. Thus, a weaker financial constraint from the part of the buyer may in some cases be associated with efficiency loses.

Related Literature The early literature on auctions and selling mechanisms with financial constraints considered public budgets. Laffont and Robert (1996) study single-item optimal auction for n bidders whose valuations are private information but budgets are common knowledge and identical — see Maskin (2000) and Benoît and Krishna (2001). Malakhov and Vohra (2008) consider the design of optimal single-item auctions for two bidders, where one bidder has a publicly-known budget constraint while the other has no budget constraint.

Our work is more closely related to the recent literature with privately known budgets. Che and Gale (2000) characterize the single-item optimal auction for a single bidder who is endowed with private valuations and private budgets. Bhattacharya et al. (2010) show that with an infinitely divisible good, a bidder cannot improve her utility by reporting a budget smaller than her actual one. See Hafalir et al. (2012) for related results. Pai and Vohra (2014) characterize the single-item optimal auction for n bidders whose valuations and budgets are private information. They show how to extend Myerson's (1981) insights using the pooling (i.e., treating certain bids alike) technique of Laffont and Robert (1996). Che et al. (2013a) show that a Bayesian incentive compatible mechanism that randomly allocates the good to budgeted agents and allows for resale welfare dominates a competitive market mechanism (or random allocation without resale). Recently, Devanur and Weinberg (2017) consider revenue-maximizing mechanisms in a single-item, single-buyer setting with arbitrary (and possibly correlated) type distribution, and Richter (2016) analyzes revenue- and welfare-maximizing mechanisms to sell a divisible good to a continuum of budget constrained agents who have independently distributed private values and budgets.

There are fewer papers that, like ours, deal with selling mechanisms or auctions in multi-item environments. Daskalakis et al. (2015) consider a revenue-maximizing seller with m items for sale to n bidders with additive valuations and private budgets. Assuming independence of budgets and valuations, they provide a mechanism that is a 3-approximation with respect to all Bayesian incentive compatible, ex-post individually rational, and ex-post budget respecting mechanisms.

Finally, we draw on graph-theoretic interpretations of incentive compatibility.[8] In single dimensional type space environments (such as single-item auctions), Myerson (1981) shows that weak monotonicity is sufficient and necessary for any allocation function to be implementable. For multi-item auctions where players have loosely correlated values for distinct subsets of items, this is no longer the case and one has to rely on the characterization of incentive compatibility via cyclic monotonicity, as in Rochet (1987). While cyclic monotonicity is a more complex condition to work with than weak monotonicity, several recent studies characterize multidimensional environments for which weak monotonicity implies cyclic monotonicity.[9] Our interpretation of the incentive constraints considers a novel flow network approach that also takes care of the budget feasibility constraints. As far as we know, the only precedent appears in Malakhov and Vohra (2008). There are several important differences between their work and ours. Critically, because they consider a setting with a public budget

[8]Recent work that exploits the graph-interpretation of incentive compatibility in different contexts includes Gui et al. (2004), Che et al. (2013b) and Echenique et al. (2013). See also Vohra (2011) and references therein.

[9]See for example Bikhchandani et al. (2006), Saks and Yu (2005), Ashlagi et al. (2010), Carroll (2012), Archer and Kleinberg (2014), Carbajal and Müller (2015), and Lavi and Swamy (2009), among others.

constraint, they do not distinguish between maximal and minimal capacities (equivalently, between restricted and unrestricted incremental values) but employ the standard length based solely on the incentive constraints.

2 Framework

We consider an environment where a financially constrained buyer (agent) interacts with a seller (principal). Our results can be extended to multi-buyer environments when dominant strategy (prior-free) incentive compatibility is the solution concept.

There is a finite set $\mathcal{A}$ of potentially different items (alternatives) that can be sold by the risk-neutral seller.[10] We normalize the cost of each of them to be equal to zero. The buyer has a non-negative, private value $v(a)$ for each item $a \in \mathcal{A}$. A *valuation* v for the buyer is thus a mapping $v\colon \mathcal{A} \to \mathbb{R}_+$. Let $\mathcal{V}$ be the bounded set of all private valuations. Notice our setting encompasses single-unit, multi-unit and multi-object (reduced) auctions as special cases. For instance, to accommodate a single-item auction we let $\mathcal{A}$ be a (finite) subset of the unit interval, and $v(a) = \nu \cdot a$ be the value associated to obtaining the item with probability a, for some $\nu \in \mathbb{R}_+$, and so forth. Multi-unit or multi-object auctions with additive valuations are similarly handled. An advantage of our approach is that it can also accommodate more complex classes of valuations; e.g., preferences over gross substitutes or valuations that exhibit complementarities among items.[11]

In addition to her private valuation, the buyer has a non-negative, privately observed *budget* B, which acts as a financial constraint on purchasing choices. Let $\mathcal{B} \subseteq \mathbb{R}_+$ denote the bounded set of all possible private budgets.[12] This specifies a multi-dimensional mechanism design setting with private valuations and private budgets. We refer to the pair (v, B) as the buyer's *type*, and denote the set of possible types by $\mathcal{T} \subseteq \mathcal{V} \times \mathcal{B}$. As is common in the literature, our model considers hard financial constraints: the buyer can never pay beyond her budget. To be specific, the utility of a buyer with type (v, B) when buying item a at a price p is $v(a) - p$ as long as $p \leq B$, and negative infinity if $p > B$.

In this setting, a direct mechanism $\{f, \rho\}$ consists of an allocation function $f\colon \mathcal{T} \to \mathcal{A}$ and a payment function $\rho\colon \mathcal{T} \to \mathbb{R}$. Without loss of generality, we treat f as a surjection — otherwise the alternative set $\mathcal{A}$ can be condensed to be the range of f. The direct mechanism $\{f, \rho\}$ is called *budget feasible for the buyer* if no payment exceeds her budget; i.e.,

$$\rho(v, B) \ \leq \ B, \qquad \text{for each type } (v, B) \text{ in } \mathcal{T}. \tag{BF}$$

It is called *incentive compatible* if any affordable deviation from truth-telling is not profitable for the buyer; i.e., if it is the case that

$$v(f(v, B)) \,-\, \rho(v, B) \ \geq \ v(f(v', B')) \,-\, \rho(v', B'), \tag{IC}$$

for all (v, B) and (v', B') in $\mathcal{T}$ such that $\rho(v', B') \leq B$. The mechanism $\{f, \rho\}$ is said to *generate no deficit for the seller* if it is the case that

$$\rho(v, B) \ \geq \ 0, \qquad \text{for all types } (v, B) \text{ in } \mathcal{T}. \tag{ND}$$

[10] This is for simplicity of exposition alone. Our results go through if $\mathcal{A}$ is infinite.

[11] In Section 5 we present two applications of our framework to revenue maximization.

[12] If the budget is publicly known, then $\mathcal{B}$ is a singleton.

Finally, $\{f, \rho\}$ is said to be *individually rational* if it never requires the buyer to pay more than her value for an alternative; i.e.,

$$v(f(v,B)) - \rho(v,B) \geq 0, \qquad \text{for all types } (v,B) \text{ in } \mathcal{T}. \tag{IR}$$

In the mechanism design literature without financial restrictions on the part of the buyer, all informational constraints are captured by incentive compatibility. When the buyer is financially constrained, it is reasonable to include the additional budgetary restriction explicitly as a requirement for implementability. Otherwise one can trivially implement any allocation function by ensuring a negative infinity payoff irrespective of the buyer's report.[13]

To focus on non-trivial implementable mechanisms, we henceforth consider incentive compatibility together with budget feasibility in our notion of implementability. Requiring (IC) and (BF) without the no deficit condition is however pointless, as the seller could subsidize the buyer's consumption choices. Thus, in our definition of implementability given below we take all these concerns explicitly into account. The (ex-post) zero deficit requirement is just a normalization —in other contexts, one could think of an ex-ante limit for deficits that a mechanism can run (in expectation).

Definition 1. In a setting with private budgets, an allocation function $f : \mathcal{T} \to \mathcal{A}$ is said to be *implementable without deficits* if there exists a payment function $\rho : \mathcal{T} \to \mathcal{A}$ such that the direct mechanism $\{f, \rho\}$ is incentive compatible (IC), budget feasible for the buyer (BF), and generates no deficit for the seller (ND).

Our results can be extended to include a participation constraint from the part of the buyer. In such settings, we assume that not transacting with the seller provides the buyer with zero utility, irrespective of her type.

Definition 2. In a setting with private budgets, an allocation function f is said to be *acceptable* if there is a payment function p such that the direct mechanism $\{f, \rho\}$ is implementable without deficits and individually rational for the buyer (IR).

Because the type space is multi-dimensional, in our setting it is quite difficult to rely on envelope techniques to express any incentive compatible payment function. Instead, we employ the next result which has been stated before in the literature —e.g., Che and Gale (2000), Borgs et al. (2005)— to express payments in terms of non-linear prices. (We include its proof for completeness.) Henceforth, we let $f^{-1}(a) \subseteq \mathcal{T}$ denote the subset of types that are assigned item a under the allocation function f. Notice $f^{-1}(a) \neq \emptyset$ for all $a \in \mathcal{A}$ (recall f is surjective).

Lemma 1 (Taxation Principle)**.** *Let $\{f, \rho\}$ be an incentive compatible and budget feasible mechanism in a setting with private valuations and private budgets. Then for every $a \in \mathcal{A}$, there exists a price $p(a) \in \mathbb{R}$ such that*

$$p(a) = \rho(v,B), \qquad \textit{for all types } (v,B) \in f^{-1}(a).$$

Proof. Suppose to obtain a contradiction that types $(v,B), (v',B')$ belong to $f^{-1}(a)$ but instead $\rho(v,B) > \rho(v',B')$. Since $\{f,\rho\}$ is budget feasible, we have that $\rho(v',B') < B$ and thus (v',B')

[13]Indeed, any direct mechanism $\{f, \rho\}$ is trivially implementable if, for all $(v,B) \in \mathcal{T}$, one has that $\rho(v,B) > \sup \mathcal{B}$ (recall $\mathcal{B}$ is bounded).

is an affordable deviation for type (v, B). Immediately,

$$v(a) - \rho(v, B) < v(a) - \rho(v', B').$$

This expression implies that (v', B') is a profitable and affordable deviation for type (v, B), contradicting (IC). $\square$

In what follows, we write incentive compatible, budget feasible *selling mechanisms* as $\{f, p\}$, where $f: \mathcal{T} \to \mathcal{A}$ is an allocation function and $p: \mathcal{A} \to \mathbb{R}$ is a price function. The rest of the conditions are easily accommodated to selling mechanisms specified via non-linear prices. Say that the selling mechanism $\{f, p\}$ is implementable without deficits if it is incentive compatible (IC), budget feasible for the buyer (BF), and generates no deficit for the seller (ND). Similarly, $\{f, p\}$ is said to be acceptable if it is implementable without deficits and individually rational for the buyer (IR).

3 Implementable Selling Mechanisms

Fix an allocation function $f: \mathcal{T} \to \mathcal{A}$ that the seller wants to use to allocate items among the different buyer's types. Before considering revenue maximization or any other goal, the seller ought to answer two questions. First, how can one be sure whether or not there exists a price function that implements f without deficits? Second, what are the properties of any such price function? In this section we address the first question and provide necessary and sufficient conditions to characterize selling mechanisms that are implementable without generating deficits for the seller. Our proofs are constructive: we define a novel allocation network associated to f and exploit its properties to determine whether f can be implemented by the seller. We explore the second question in Section 4.

3.1 Incremental values

Given f, for any alternative $a \in \mathcal{A}$ let $f^{-1}(a)|\mathcal{B}$ denote the projection of $f^{-1}(a)$ to $\mathcal{B}$. Define

$$\beta(a) \equiv \inf \left\{B : B \in f^{-1}(a)|\mathcal{B}\right\}$$

as the minimal budget that the buyer is asked to report to obtain item a under f. We refer to $\beta(a)$ as the *budget level* for a. Clearly, $0 \leq \beta(a) < \infty$. Observe that $\beta(a)$ serves as an upper bound on any budget feasible price that the buyer may be asked to pay for a under any selling mechanism $\{f, p\}$ that is implementable without deficits.

For every pair of items $a, a' \in \mathcal{A}$, $a \neq a'$, define the buyer's *unrestricted incremental value* between a and a' by

$$\delta(a, a') \equiv \inf \left\{v(a) - v(a') : (v, B) \in f^{-1}(a)\right\}. \tag{1}$$

Similarly, define the *restricted incremental value* between a and a' by

$$\delta^r(a, a') \equiv \inf \left\{v(a) - v(a') : (v, B) \in f^{-1}(a), B \geq \beta(a')\right\}. \tag{2}$$

The unrestricted incremental value $\delta(a, a')$ is the minimal value difference that the buyer enjoys from receiving a instead of a' under the allocation function f. The negative of $\delta(a, a')$

can be understood as the hypothetical maximal value gained by the buyer when, ignoring the recommendation of the seller, she purchases a' instead of a. Without budget constraints, $\delta(a,a')$ informs the seller's construction of implementable prices by specifying the value of the most profitable deviation from the buyer's perspective. In the presence of budget constraints, however, a deviation to purchasing a' instead of a may not be affordable for all types in $f^{-1}(a)$. On the other hand, $\delta^r(a,a')$ is the minimal value difference from receiving a instead of a' for those types who are able to *afford* item a'. This subtle difference plays a crucial role in obtaining our results. Clearly, for all items $a, a' \in \mathcal{A}$, $a \neq a'$,

$$-\infty \ < \ \delta(a,a') \ \leq \ \delta^r(a,a') \ \leq \ \infty.$$

When the buyer's budget is publicly known (in which case $\mathcal{B}$ is a singleton), equality holds between restricted and unrestricted incremental values.

It is important to understand that under private budgets any wedge between the unrestricted and the restricted incremental values depends on the allocation decisions under f. In this case, equality holds if a' is affordable for the buyer whenever alternative a is affordable, as would be when $\beta(a') \leq \beta(a)$. Instead, when the budget level for a' is strictly higher than the budget level for a, equality between the unrestricted and the restricted incremental values hold when the allocation function f does not restrict purchases of a according to budget size. This is illustrated in Figure 1a, where the horizontal axis represents the buyer's valuation difference between a and a', the vertical axis represents the buyer's financial constraints B, and each point in the plane represents a type (v,B) that chooses a under f. Stated differently, Figure 1a represents a situation where the allocation function f does not reserve a solely for buyers with smaller budgets.

The opposite situation is illustrated in Figure 1b. There, types with large budgets are prevented by f from obtaining item a: $f(v,B) \neq a$ for every type (v,B) with $B \geq \beta(a') > \beta(a)$. Here one has $\delta^r(a,a') = +\infty$, as now the restricted incremental value $\delta^r(a,a')$ is the infimum over an empty set. Finally, a situation could arise where the allocation function f partially restricts access to a via budgets in the following sense: a buyer with a high budget can purchase item a under f, but only if her valuation for a is sufficiently large relative to her valuation for a'. For such an allocation function, the difference between $\delta(a,a')$ and $\delta^r(a,a')$ is strict, and in addition $\delta^r(a,a') < +\infty$. This is represented in Figure 1c. In other words, the allocation function f does not discriminate against buyers with large budgets as long as their valuation for item a is large.

Incremental values provide an upper bound on incentive compatible price differences.

Proposition 1. *Let $\{f,p\}$ be a budget feasible selling mechanism.*

(a) If $\{f,p\}$ is incentive compatible, then $\delta^r(a,a') \geq p(a) - p(a')$ for all items $a, a' \in \mathcal{A}$.

(b) If $\delta(a,a') \geq p(a) - p(a')$ for all $a, a' \in \mathcal{A}$, then $\{f,p\}$ is incentive compatible.

(c) If the buyer's financial constraint is publicly known, then $\{f,p\}$ is incentive compatible if, and only if, for all $a, a' \in \mathcal{A}$ one has

$$+\infty \ > \ \delta(a,a') \ \geq \ p(a) \ - \ p(a') \ \geq \ -\delta(a',a) \ > \ -\infty.$$

Proof. (a) Let $\{f,p\}$ be incentive compatible and budget feasible for the buyer. Let a, a' be two alternatives such that $\delta^r(a,a') \in \mathbb{R}$. For arbitrarily small $\epsilon > 0$, there exists a type

$(v, B) \in f^{-1}(a)$ such that $B \geq \beta(a')$ and $v(a) - v(a') \leq \delta^r(a, a') + \epsilon$. Since $p(a') \leq \beta(a')$ by (BF), the buyer can afford to purchase a' when her type is (v, B). The selling mechanism satisfies (IC), therefore it follows that

$$p(a) - p(a') \leq v(a) - v(a') \leq \delta^r(a, a') + \epsilon.$$

This last expression holds for all $\epsilon > 0$ sufficiently small, thus $\delta^r(a, a') \geq p(a) - p(a')$, as required. The remaining cases are easily dealt with. Clearly, the claim trivially holds if $\delta^r(a, a') = \infty$. In addition, f is incentive compatible and therefore it cannot be the case that $\delta^r(a, a') = -\infty$ (recall that f is onto).

(b) Suppose that $\{f, p\}$ is not incentive compatible. When (IC) is violated, this implies that there exist two alternatives $a, a' \in \mathcal{A}$ and a type $(v, B) \in f^{-1}(a)$ such that $p(a') \leq B$ and $-\infty < v(a) - p(a) < v(a') - p(a')$, where the first inequality holds as the selling mechanism satisfies (BF). Immediately, $\delta(a, a') \leq v(a) - v(a') < p(a) - p(a')$.

(c) The claim follows from parts (a) and (b), noticing that when the budget is public one has $\delta^r(a, a') = \delta(a, a')$, for all items a, a' in $\mathcal{A}$. □

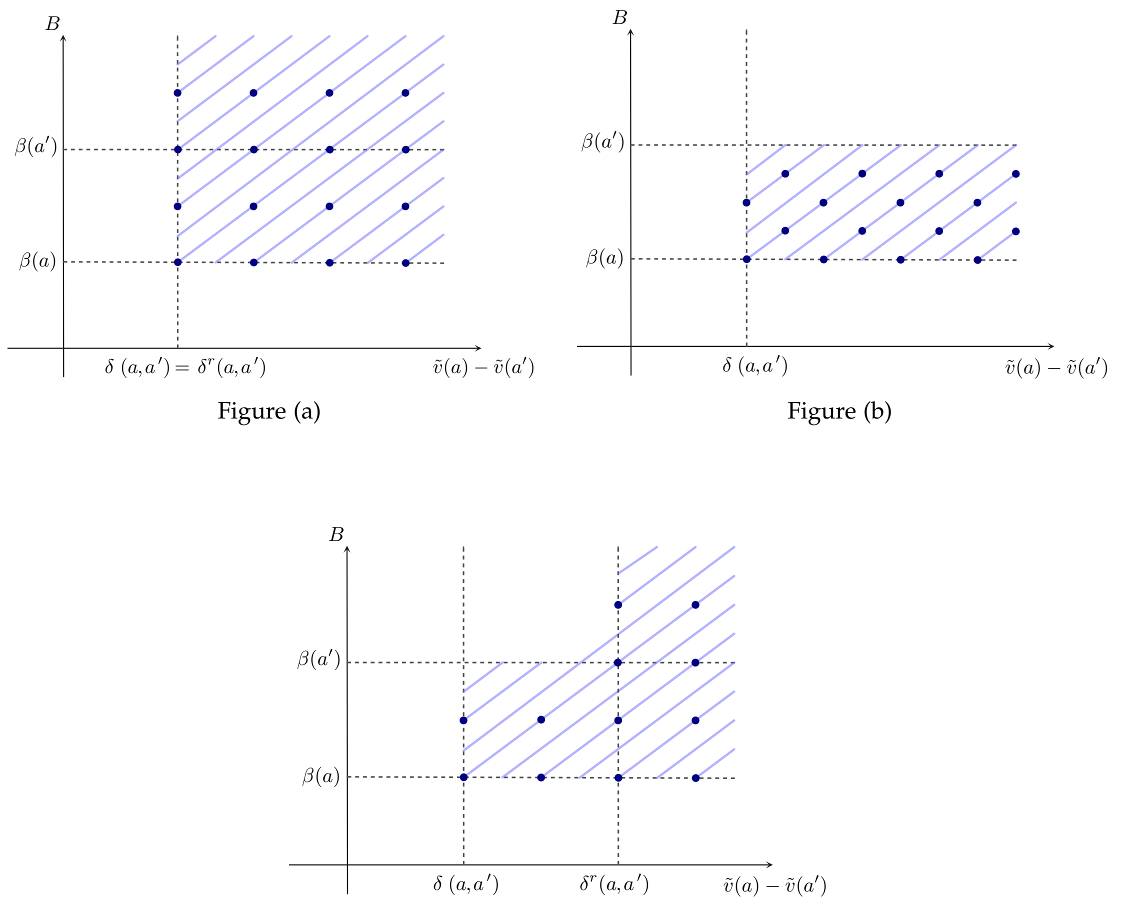

Figure (a)

Figure (b)

Figure (c)

Figure 1: Unrestricted and restricted incremental values between a and a'.

3.2 The allocation network

In multi-dimensional mechanism design settings without financial constraints, a standard approach to characterize incentive compatibility is to construct a complete directed graph associated to f, whose nodes are alternatives in $\mathcal{A}$, and endow it with a length for each edge $e = (a, a')$ that is equal to the unrestricted incremental value $\delta(a, a')$.[14] Incentive compatibility is then equivalent to the fact that no cycle in this graph is of negative length (Rochet, 1987).

There are two related problems in translating this approach to characterize implementability without deficits when the buyer faces financial constraints. First, with private budgets, the gap between parts (a) and (b) of Proposition 1 cannot always be bridged. More specifically, not every pricing function p associated to f and satisfying the condition

$$\delta^r(a, a') \;\geq\; p(a) - p(a') \;\geq\; -\delta^r(a', a)$$

is incentive compatible. Moreover, part (a) of Proposition 1 may no longer be valid if $\delta^r(a, a')$ is replaced by $\delta(a, a')$ —we provide examples to illustrate this point later on. As a result, the *if and only if* connection between incentive compatibility and the standard formulation of cyclic monotonicity does not carry over to private budget settings.

The second problem is that, even when the buyer's budgetary restriction is publicly known, and thus restricted and unrestricted incremental values coincide —i.e., $\delta^r(a, a') = \delta(a, a')$ for every pair of items in $\mathcal{A}$—, using incremental values to define the length in the allocation graph does not rule out violating some other condition of implementability without deficits. Example 1 below illustrates this point.

To overcome these hurdles, we pursue a flow network approach and introduce a novel allocation network whose nodes consist of elements of $\mathcal{A}$, in addition to a fictional alternative a_0 that constitutes the sink of the allocation network. (When considering acceptable mechanisms, a_0 can be interpreted as the outside option of the buyer.) We endow the edges of the network with minimal and maximal capacities, instead of a length. These capacities play an important role in the determination of incentive prices —a connection that we explore in Section 4.

Definition 3. The *allocation network* $H = (N, E)$ associated to $f \colon \mathcal{T} \to \mathcal{A}$ is composed of a set nodes $N = \mathcal{A} \cup \{a_0\}$ and a set of directed edges $E = E_1 \cup E_2 \cup E_3$, where

$$E_1 \;=\; \mathcal{A} \times \mathcal{A}, \quad E_2 \;=\; \mathcal{A} \times \{a_0\}, \quad \text{and} \quad E_3 \;=\; \{a_0\} \times \mathcal{A}.$$

Every edge $e = (x, y) \in E$ is endowed with a *minimal capacity* $\underline{\kappa}(x, y)$ and a *maximal capacity* $\overline{\kappa}(x, y)$ defined as follows:

- $\underline{\kappa}(a, a') \;:=\; \delta(a, a') \;\leq\; \delta^r(a, a') \;=:\; \overline{\kappa}(a, a')$, for all $(a, a') \in E_1$;
- $\underline{\kappa}(a, a_0) \;:=\; \beta(a) \;=:\; \overline{\kappa}(a, a_0)$, for all $(a, a_0) \in E_2$;
- $\underline{\kappa}(a_0, a) \;:=\; 0 \;=:\; \overline{\kappa}(a_0, a)$, for all $(a_0, a) \in E_3$.

In the standard formulation of cyclic monotonicity, $E_1 = \mathcal{A} \times \mathcal{A}$ constitutes the entire graph. Our allocation network $H = (N, E)$ considers, in addition, directed edges connecting

[14] This tradition goes back to Rochet (1987). See also Bikhchandani et al. (2006), Heydenreich et al. (2009), Ashlagi et al. (2010), Carbajal and Müller (2015), and Edelman and Weymark (2018) among others.

each $a \in \mathcal{A}$ to the sink a_0 (edges in E_2), and directed edges from the sink a_0 to each a in $\mathcal{A}$ (edges in E_3). Figure 2 presents an illustration for the three-item case. The numbers $r \backslash s$ on top of the edges represent the minimal and maximal capacities, respectively.

Some additional notation is required before we state our results. As usual, a path $P = \{x = x_1, \ldots, x_n = y\}$ between nodes x and y in the allocation network H is a finite collection of nodes such that $e_i = (x_i, x_{i+1})$ is an edge in E, for all $i = 1, \ldots, n$. A cycle C in H is a path $\{x = x_1, \ldots, x_{n+1} = x\}$ whose initial and terminal node coincide. The minimal (maximal) capacity of P is the sum of minimal (maximal) capacities of its component edges:

$$\underline{\kappa}(P) = \sum_{i=1}^{n-1} \underline{\kappa}(x_i, x_{i+1}) \quad \text{and} \quad \overline{\kappa}(P) = \sum_{i=1}^{n-1} \overline{\kappa}(x_i, x_{i+1}).$$

The *minimal charge* $\underline{c}(x, y)$ between nodes x and y in the allocation network $H = (N, E)$ is the minimal capacity of the path with lowest minimal capacity between x and y. Similarly, the *maximal charge* $\overline{c}(x, y)$ between x and y in $H = (N, E)$ is the maximal capacity of the path with lowest maximal capacity connecting x and y.

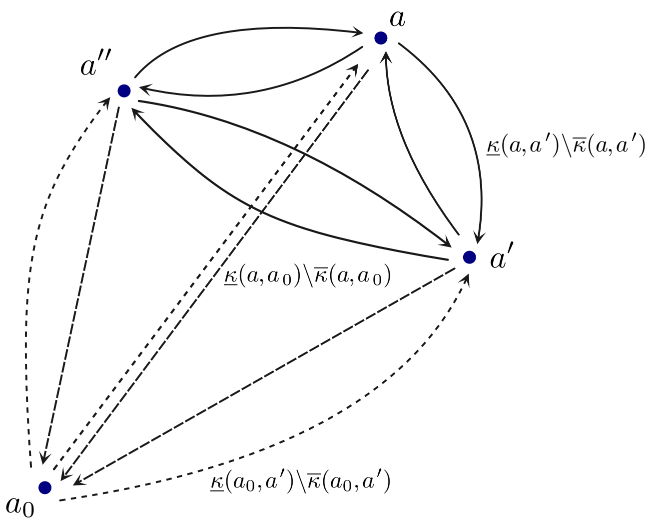

Figure 2: The allocation network H.

3.3 Implementability without deficits

A key aspect of the allocation network $H = (N, E)$ is that it captures the constraints missing in the standard formulation of incentive compatibility via cyclic monotonicity, even when budgets constraints are public information.

Example 1. There are only two possible items to allocate, $\mathcal{A}_1 = \{a, a'\}$, a singleton budget set $\mathcal{B}_1 = \{5\}$ for the financially constrained buyer, and two private valuations, $\mathcal{V}_1 = \{v, v'\}$, where

	v	v'
a	20	10
a'	10	0

Let f_1 be such that $f_1(v, 5) = a$ and $f_1(v', 5) = a'$. Since the financial constraint of the buyer is public, it is immediate to verify that

$$\underline{\kappa}(a,a') \;=\; \delta_1(a,a') \;=\; 10 \;=\; \delta_1^r(a,a') \;=\; \overline{\kappa}(a,a') \quad \text{and}$$
$$\underline{\kappa}(a'a) \;=\; \delta_1(a',a) \;=\; -10 \;=\; \delta_1^r(a',a) \;=\; \overline{\kappa}(a',a).$$

Thus, the unique cycle between alternatives a and a' has zero (maximal and minimal) capacity. However, f_1 is not implementable without deficits. Indeed, writing down the incentive constraints obtains

$$10 \;=\; v(a) - v(a') \;\geq\; p_1(a) - p_1(a') \;\geq\; v'(a) - v'(a') \;=\; 10.$$

Note that $p_1(a)$ must be less than 5 by (BF) whereas $p_1(a')$ is required to non-negative by (ND). In light of the above expression, these two conditions cannot be satisfied simultaneously. Consider now the allocation network H_1 associated with f_1. Here the cycle $C = \{(a',a),(a,a_0),(a_0,a')\}$ has a (maximal and minimal) capacity equal to -5. ◇

From Example 1 one conjectures that implementability without deficits is linked to cycles in H having non-negative capacities. Indeed, our sufficient and necessary conditions for implementability without deficits in private budget settings are based on properties of cycles in the allocation network H.

Proposition 2 (Necessity)**.** *If the allocation function f is implementable without deficits, then the corresponding allocation network $H = (N,E)$ contains no cycle of negative maximal capacity.*

Proof. Let p be a pricing function such that the selling mechanism $\{f,p\}$ satisfies (IC), (BF) and (ND). We divide the proof in two steps. The first is analogous to the original arguments in Rochet (1987), albeit considering financial restrictions in the definition of the maximal capacity. The second one uses the structure of the allocation network H.

Step 1: If C is a cycle in E_1, then it has non-negative maximal capacity. Suppose by contradiction that there is a finite cycle C in E_1 whose maximal capacity is negative:

$$\overline{\kappa}(C) \;=\; \sum_{(a,a')\in C} \overline{\kappa}(a,a') \;<\; 0.$$

Since each edge $(a,a') \in C$ belongs to E_1, by part (a) of Proposition 1, we have that

$$\overline{\kappa}(a,a') \;=\; \delta^r(a,a') \;\geq\; p(a) - p(a').$$

Adding these inequalities leads to $\overline{\kappa}(C) \geq 0$, which is a contradiction.

Step 2: If C is an arbitrary cycle in $H = (N,E)$, then C has non-negative maximal capacity. Assume on the contrary that C is a cycle with negative maximal capacity in H. Consider a partition of C into

$$C^+ \;=\; \{(a,a') \in C : \, (a,a') \in E_1\} \quad \text{and} \quad C^- \;=\; C \setminus C^+.$$

If $C^- = \emptyset$, then the result follows immediately from the previous step. If, on the other hand, $C^+ = \emptyset$, then immediately we have that $\overline{\kappa}(C) \geq 0$, which is a contradiction. Indeed, in this case each edge e in C has maximal capacity $\overline{\kappa}(a,a_0) = \beta(a) \geq 0$, or $\overline{\kappa}(a_0,a) = 0$.

So, assume that both C^+ and C^- are non-empty sets. Without loss of generality, assume that all edges in C^+ are consecutive. Thus, C^+ is a path with first node a and last node a',

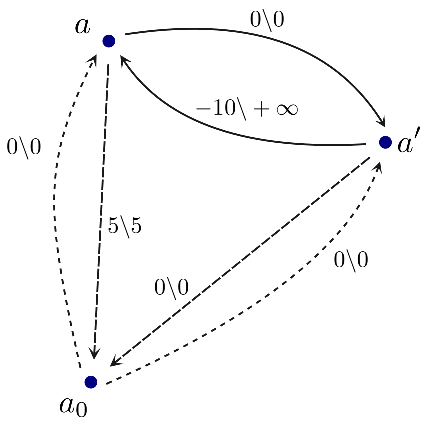

Figure 3: Allocation network for Example 2.

and edges (a', a_0), (a_0, a) in C^- completing the cycle. If this is not the case, then C can be split into two or more sub-cycles with similar structure, at least one of which will have negative maximal capacity. Now notice that

$$0 > \overline{\kappa}(C) = \sum_{e \in C^+} \delta^r(e) + \beta(a').$$

Using again part (a) of Proposition 1, from the above expression it follows that

$$0 > p(a) - p(a') + \beta(a').$$

Since the selling mechanism satisfies (ND), we have $p(a)$ is non-negative, and therefore $p(a') = \beta(a') + \epsilon$, for some $\epsilon > 0$. But f is onto, and thus there exists a type $(v, B) \in f^{-1}(a')$ such that $\beta(a') \geq B - \epsilon/2$. Therefore, $p(a') > B$, contradicting (BF) because the buyer's budget constraint is violated, which is impossible. □

The next example shows that the reverse direction of Proposition 2 is incorrect.

Example 2. The alternative set contains just two items, $\mathcal{A}_2 = \{a, a'\}$. The buyer has two possible private budgets, $\mathcal{B}_2 = \{0, 5\}$, and two possible private values $\mathcal{V}_2 = \{v, v'\}$, where

	v	v'
a	20	10
a'	10	10

Let f_2 be defined on $\mathcal{T}_2 = \mathcal{V}_2 \times \mathcal{B}_2$ by $f_2(v, 5) = f_2(v', 5) = a$ and $f_2(v, 0) = f_2(v', 0) = a'$. Figure 3 represents the allocation graph H_2 associated with f_2. Observe that no cycle in H_2 has negative maximal capacity. Despite this, we claim that f_2 is not implementable without deficits. To see why, observe that $p_2(a')$ must be 0 for (BF) and (ND) to be satisfied. From the incentive constraint for type $(v', 5)$ and using (ND) again, we conclude that $p(a) = 0$ as well. But then the buyer with type $(v, 0)$ would benefit from misreporting $(v, 5)$. ◇

In Example 2, while the allocation network associated with the allocation function f_2 has no cycle of negative maximal capacity, the cycle $\{(a', a), (a, a_0), (a_0, a')\}$ has negative minimal capacity. One conjectures that cyclic monotonicity on H with respect to minimal capacities is a sufficient condition for implementability without deficits. This observation leads to our second main result, the proof of which provides an algorithm to construct a price function directly from the allocation network H.

Proposition 3 (Sufficiency). *If the allocation network $H = (N, E)$ associated with f contains no cycle with negative minimal capacity, then f is implementable without deficits by the price function $p\colon \mathcal{A} \to \mathbb{R}$ defined by*

$$p(a) \;=\; \underline{c}(a, a_0), \qquad \textit{for all } a \in \mathcal{A}.$$

Proof. Since $H = (N, E)$ contains no cycle of negative minimal capacity, the minimal charge between a and a_0 satisfies $\underline{c}(a, a_0) > -\infty$. We show that the price function $p(\cdot) = \underline{c}(\cdot, a_0)$ implements f without deficits.

First notice that there is a direct link between a and a_0 with minimal capacity $\underline{\kappa}(a, a_0) = \beta(a)$. Hence, for arbitrary item $a \in \mathcal{A}$ we have

$$p(a) \;=\; \underline{c}(a, a_0) \;\le\; \underline{\kappa}(a, a_0) \;=\; \beta(a).$$

This shows that the price function satisfies (BF). To show (ND), recall that $\underline{\kappa}(a_0, a) = 0$ and so

$$p(a) \;=\; \underline{c}(a, a_0) \,+\, \underline{\kappa}(a_0, a) \;\ge\; 0,$$

where the last inequality is validated by the fact that all cycles in the network H have non-negative minimal capacity.

It remains to show that the selling mechanism $\{f, p\}$ satisfies (IC). Suppose to the contrary that there exist types (v, B) and (v', B') such that $f(v, B) = a$, $f(v', B') = a'$ and $v(a) - p(a) < v(a') - p(a')$, where $p(a') \le B$ so that item a' is affordable for type (v, B). Rearranging we have that

$$\begin{aligned}
\underline{\kappa}(a, a') \,+\, \underline{c}(a', a_0) \;&=\; \delta(a, a') \,+\, p(a') \\
&\le\; v(a) \,-\, v(a') \,+\, p(a') \;<\; p(a) \;=\; \underline{c}(a, a_0).
\end{aligned}$$

The left-hand side represents the minimal capacity of a path from a to a_0 through a', which is strictly smaller than the minimal capacity $\underline{c}(a, a_0)$ of the shortest path from a to a_0. This is clearly a contradiction. □

The fact that H admits no cycle with negative minimal capacity is not a necessary condition for implementation without deficits. The following example serves as illustration. Together with Example 2, these examples show that our results are tight.

Example 3. Consider a setting with two possible outcomes, $\mathcal{A}_3 = \{a, a'\}$, two possible private budgets $\mathcal{B}_3 = \{0, 5\}$, and a single valuation $\mathcal{V}_3 = \{v\}$ for the buyer, where $v(a) = 10$ and $v(a') = 0$. Let f_3 be an allocation function defined on $\mathcal{T}_3 = \mathcal{V}_3 \times \mathcal{B}_3$ by $f_3(v, 5) = a$ and $f_3(v, 0) = a'$. The allocation network associated with f_3 is similar to the allocation network in Figure 3, with the exception of the minimal and maximal capacities between nodes a and a'; here one has $\underline{\kappa}(a, a') = \overline{\kappa}(a, a') = 10$. The cycle $\{(a', a), (a, a_0), (a_0, a')\}$ has negative minimal capacity. But it is easy to see that f_3 is implementable without deficits, for instance by using the pricing function $p_3(a) = 5$ and $p_3(a') = 0$. ◇

When minimal and maximal capacities coincide for all the edges in the network $H = (N, E)$, an exact characterization of implementability without deficits is readily available. This characterization applies, in particular, for settings with financially constrained buyers where budgets are publicly known.[15]

Corollary 1. *Given* $f\colon \mathcal{T} \to \mathcal{A}$*, suppose that* $\underline{\kappa}(a,a') = \overline{\kappa}(a,a')$ *for all* $(a,a') \in E$*. Then* f *is implementable without deficits if, and only if, the allocation network* $H = (N,E)$ *contains no cycles of negative capacity.*

Proof. The result follows immediately from Proposition 3 and Proposition 2. □

3.4 Acceptable mechanisms

Our results in the previous section provide a detailed picture of selling mechanisms that are implementable without deficits. By modifying the definitions of the capacities of edges in $E_2 = \mathcal{A} \times \{a_0\}$ in the allocation network H, we can extend these to acceptable mechanisms. A bit more notation is required. For any a in $\mathcal{A}$, let $f^{-1}(a)|\mathcal{V}(a)$ denote the projection of $f^{-1}(a)$ to $\mathcal{V}(a)$. Define

$$\omega(a) \;\equiv\; \inf\left\{v(a) : v(a) \in f^{-1}(a)|\mathcal{V}(a)\right\}.$$

Here $\omega(a)$ is the minimal reported valuation required to obtain item a under the allocation function f. As with the budget level $\beta(a)$, we notice that $0 \le \omega(a) < \infty$ and point out that $\omega(a)$ serves as an upper bound on any individually rational price that the buyer may be asked to pay for a under the selling mechanism $\{f,p\}$.

Given $f\colon \mathcal{T} \to \mathcal{A}$, the allocation network $\tilde{H}$ is defined in the same way as H, with the following exception: the (maximal and minimal) capacities for all $(a,a_0) \in E_2$ are now given by

$$\underline{\kappa}(a,a_0) \;:=\; \min\{\beta(a),\omega(a)\} \;=:\; \overline{\kappa}(a,a_0).$$

We obtain the following results, the proofs of which follow the lines of their counterparts for implementable mechanism without deficits.

Proposition 4. *If the selling mechanism* $\{f,p\}$ *is acceptable, then the allocation network* $\tilde{H}$ *contains no cycle of negative maximal capacity.*

Proposition 5. *If the allocation network* $\tilde{H}$ *associated with* f *contains no cycle of negative minimal capacity, then there exists a pricing function* $p\colon \mathcal{A} \to \mathbb{R}$ *such that the selling mechanism* $\{f,p\}$ *is acceptable.*

4 The Structure of Prices for Selling Mechanisms

In this section we provide additional results on prices for allocation functions that are implementable without deficits, based on the properties of the allocation network $H = (N,E)$. Throughout this section, we assume that $f\colon \mathcal{T} \to \mathcal{A}$ is implementable without deficits. From Proposition 2, it follows that the allocation network H has no cycle of negative maximal capacity. Our first observation on the structure of prices for selling mechanisms is that they are bounded above by maximal charges.

[15] Recall the publicly known budget setting is a special case of the private budget setting where $\mathcal{B}$ is a singleton.

Proposition 6 (Price Bounds)**.** *If the pricing function p implements f without deficits, then*

$$p(a) \;\leq\; \bar{c}(a, a_0), \qquad \textit{for all } a \in \mathcal{A}.$$

Proof. Since f is implementable without deficit, the allocation network H contains no cycle of negative maximal capacity. Let $P^* = \{a, a_1, a_2, \ldots, a_k, a_0\}$ be a path with lowest maximal capacity from node a to node a_0 in $H = (N, E)$, so that the maximal charge $\bar{c}(a, a_0)$ between a and a_0 equals the maximal capacity of P^*. If the selling mechanism $\{f, p\}$ is implementable without deficits, using Proposition 1(a) one obtains

$$\begin{aligned}\bar{c}(a, a_0) \;&=\; \bar{\kappa}(a, a_1) \,+\, \ldots \,+\, \bar{\kappa}(a_{k-1}, a_k) \,+\, \bar{\kappa}(a_k, a_0)\\ &=\; \delta^r(a, a_1) \,+\, \ldots \,+\, \delta^r(a_{k-1}, a_k) \,+\, \beta(a_k)\\ &\geq\; p(a) \,-\, p(a_k) \,+\, \beta(a_k) \;\geq\; p(a).\end{aligned}$$

□

The intuition behind the upper bounds on the pricing function generalizes the construction of marginal incentive prices when there is a single divisible good to allocate. Essentially, the price of (an additional unit of) a cannot exceed the incremental value from consuming it. With several different objects, there are several possible ways of calculating incremental value differences, any of which can act as an upper bound. Thus, the maximal charge concisely expresses such requirement.

When the buyer faces no financial constraints, an immediate implication of quasi-linearity is that the convex combination of any two incentive compatible price functions preserves incentive compatibility. But the buyer's budgetary restrictions can make some of the incentive constraints inactive for a given price function and active for another one. Thus, in general, the convex combination of two price functions that implement an allocation function without deficits need not be incentive compatible, as we now illustrate.

Example 4. There are two alternatives, $\mathcal{A}_4 = \{a, a'\}$, two possible budgets, $\mathcal{B}_4 = \{4, 5\}$, and finally a single valuation for the buyer, $\mathcal{V}_4 = \{v\}$, where $v(a) = 8$ and $v(a') = 10$. The allocation function f_4 is defined on $\mathcal{T}_4 = \mathcal{V}_4 \times \mathcal{B}_4$ by $f_4(v, 4) = a$ and $f_4(v, 5) = a'$. Consider two different price functions: $p(a) = 0, p(a') = 2$, and $p'(a) = 4, p'(a') = 5$. Clearly, both p and p' satisfy (IC), (BF) and (ND). Notice that, under p', the low budget buyer cannot afford alternative a'.

Now consider a new price function $\hat{p}$ resulting from taking a convex combination of p and p'; that is $\hat{p} = \frac{1}{2}p + \frac{1}{2}p'$. At the new prices $\hat{p}(a) = 2$ and $\hat{p}(a') = 3.5$, the low budget type $(v, 4)$ has a profitable deviation that is affordable: purchase a' instead of a —the price function $\hat{p}$ violates (IC). ◇

Despite the lack of convexity in the general structure of incentive compatible prices, our next proposition shows that price functions constitute a lattice, a finding that mirrors results for selling mechanisms without financial constraints.[16] Recall that given any two real-valued functions ρ, ρ' on $\mathcal{A}$, their joint $\rho \vee \rho'$ and meet $\rho \wedge \rho'$ are defined, respectively, by

$$\rho \vee \rho'(a) \;=\; \max\{\rho(a), \rho'(a)\} \quad \text{and} \quad \rho \wedge \rho'(a) \;=\; \min\{\rho(a), \rho'(a)\}, \quad \text{for all } a \in \mathcal{A}.$$

[16] See for instance Demange et al. (1986).

Proposition 7 (Lattice Structure)**.** *Suppose the selling mechanisms $\{f, p'\}$ and $\{f, p''\}$ are implementable without deficits. Then*

$$\{f, p' \vee p''\} \qquad \text{and} \qquad \{f, p' \wedge p''\}$$

are also implementable without deficits.

Proof. Clearly, $p' \vee p''$ and $p' \wedge p''$ satisfy (BF) and (ND). Let $(v, B) \in f^{-1}(a)$ and $a' \neq a$ be any item in $\mathcal{A}$. By a symmetry argument, to establish that $p' \vee p''$ satisfies (IC), it is sufficient to consider the following 2 cases:

Case 1: $p'(a') \leq B$ and $p''(a') \leq B$. Then, since a' is affordable for type (v, B) under both price functions, we have

$$v(a) - p'(a) \geq v(a') - p'(a') \qquad \text{and} \qquad v(a) - p''(a) \geq v(a') - p''(a').$$

If $p' \vee p''(a) = p'(a)$ and $p' \vee p''(a') = p''(a')$, then one has

$$v(a) - p' \vee p''(a) = v(a) - p'(a) \geq v_(a') - p'(a') \geq v(a') - p' \vee p''(a').$$

Otherwise, if $p' \vee p''(a) = p''(a)$ and $p' \vee p''(a') = p'(a')$, then

$$v(a) - p' \vee p''(a) = v(a) - p''(a) \geq v_(a') - p''(a') \geq v(a') - p' \vee p''(a').$$

Case 2: either $p'(a') > B$ or $p''(a') > B$. Clearly, $p' \vee p''(a') > B$ is not affordable for type (v, B) at price $p' \vee p''$ and thus deviation from a to a' is impossible.

It remains to show that $p' \wedge p''$ satisfies (IC). By a symmetry argument, here it also suffices to consider the following three cases:

Case 1: $p'(a') \leq B$ and $p''(a') \leq B$. Then a' is affordable for type (v, B) under both price schemes, and thus by (IC) we have

$$v(a) - p'(a) \geq v(a') - p'(a') \qquad \text{and} \qquad v(a) - p''(a) \geq v(a') - p''(a').$$

If $p' \wedge p''(a) = p'(a)$ and $p' \wedge p''(a') = p''(a')$,

$$v(a) - p' \wedge p''(a) \geq v(a) - p''(a) \geq v_(a') - p''(a') = v(a') - p' \wedge p''(a').$$

Otherwise, if $p' \wedge p''(a) = p''(a)$ and $p' \wedge p''(a') = p'(a')$, then

$$v(a) - p' \wedge p''(a) \geq v(a) - p'(a) \geq v_(a') - p'(a') = v(a') - p' \wedge p''(a').$$

Case 2: $p'(a') > B$ and $p''(a') > B$. Clearly, $p' \wedge p''(a') > B$ and therefore the deviation from a to a' is not affordable for the type (v, B).

Case 3: $p'(a') \leq B < p''(a')$. Here, by (IC) it must be that $v(a) - p'(a) \geq v(a') - p'(a')$. Readily,

$$v(a) - p' \wedge p''(a) \geq v(a) - p'(a) \geq v_(a') - p'(a') = v(a') - p' \wedge p''(a'). \qquad \square$$

The importance of Proposition 6 and Proposition 7 is that we can use them to derive maximal (revenue-maximizing) prices compatible with any given allocation function that is implementable without deficits for the 2-item case.

Proposition 8 (Maximal Prices in the 2-Item Case)**.** *Let $\mathcal{A} = \{a, a'\}$ and suppose the allocation function $f\colon \mathcal{T} \to \mathcal{A}$ is implementable without deficits. Let p^* be a pricing function defined by*

$$p^*(a) \;=\; \bar{c}(a, a_0) \qquad \textit{and} \qquad p^*(a') \;=\; \bar{c}(a', a_0).$$

Then the selling mechanism $\{f, p^*\}$ satisfies (IC)*,* (BF) *and* (ND)*.*

Proof. Obtaining (BF) and (ND) is shown in a way similar to the proof of Proposition 3 and thus omitted. To show that the selling mechanism $\{f, p^*\}$ satisfies (IC), assume by contradiction that there exists a type $(v, B) \in f^{-1}(a)$ such that $v(a) - \bar{c}(a, a_0) < v(a') - \bar{c}(a', a_0)$ and further $\bar{c}(a', a_0) \leq B$, so that deviation to buying a' is affordable for type (v, B). We shall consider the following exhaustive cases:

Case 1: $\beta(a') \leq \beta(a)$ or $B \geq \beta(a') > \beta(a)$. Here we have that $\delta^r(a, a') \leq v(a) - v(a') < \bar{c}(a, a_0) - \bar{c}(a', a_0)$. Equivalently, $\bar{\kappa}(a, a') + \bar{c}(a', a_0) < \bar{c}(a, a_0)$, which is a contradiction to the definition of the maximal charge $\bar{c}(a, a_0)$ as the maximal capacity of the path with smallest maximal capacity between nodes a and a_0 in the allocation network H.

Case 2: $\beta(a') > B \geq \beta(a)$. Since f is implementable without deficits, there exists p such that $\{f, p\}$ satisfies (IC), (BF) and (ND). Let p be the highest possible price function. That is, for every p' that implements f without deficits, we have that $p'(a) \leq p(a)$ and $p'(a') \leq p(a')$. Notice that by Proposition 6 and Proposition 7, the price p function is well defined.

Since $p(a') \leq \bar{c}(a', a_0) \leq B$, we have

$$v(a) \;-\; \bar{c}(a, a_0) \;<\; v(a') \;-\; \bar{c}(a', a_0) \;\leq\; v(a') \;-\; p(a') \;\leq\; v(a) - p(a).$$

This implies $p(a) < \bar{c}(a, a_0) \leq \beta(a)$. Additionally, $p(a') \leq \bar{c}(a', a_0) \leq B < \beta(a')$. Now for sufficiently small $\epsilon > 0$ we have that the selling mechanism $\{f, p + \epsilon\}$ is implementable without deficits, a contradiction to the maximality of p. □

Note that Proposition 8 does not, by itself, solve the revenue maximization problem of a seller in the 2-item case. But it allows the seller to simplify the problem and focus the search on allocation functions that are implementable without deficits —maximal prices for such f are pinned down by the maximal charges of the corresponding allocation network. Since minimal charges also constitute implementable prices (see Proposition 3), the following corollary provides the structure of all implementable prices that can be extracted from the network $H = (N, E)$ associated to an allocation function in the 2-item case.

Corollary 2. *Let $\mathcal{A} = \{a, a'\}$ and suppose the network $H = (N, E)$ associated with the allocation function f contains no cycle of negative maximal or minimal capacity. Let p be any price function such that*

$$\underline{c}(a, a_0) \;\leq\; p(a) \;\leq\; \bar{c}(a, a_0) \qquad \textit{and} \qquad \underline{c}(a', a_0) \;\leq\; p(a') \;\leq\; \bar{c}(a', a_0).$$

Then the selling mechanism $\{f, p\}$ is implementable without deficits.

Proof. Fix an allocation function f and prices that satisfy the condition of the corollary. We first observe that both minimal charges $\underline{c}(a, a_0), \underline{c}(a', a_0)$ and maximal charges $\overline{c}(a, a_0), \overline{c}(a', a_0)$ implement f without deficits. It follows immediately that the price function p is budget feasible for the buyer (BF) and does not raise a deficit for the seller (ND).

To show that p satisfies (IC), without loss of generality let $\beta(a) \leq \beta(a')$, which means that item a always constitutes an affordable deviation for types that are assigned a' under the allocation rule f. This implies, by construction, that the minimal and maximal capacities coincide: $\underline{\kappa}(a', a) = \overline{\kappa}(a', a)$, and therefore $\underline{c}(a', a_0) = \overline{c}(a', a_0) = p(a')$. It follows that any type $(v', B') \in f^{-1}(a')$ does not have an incentive to deviate and purchase a, as

$$v'(a') - p(a') = v'(a') - \underline{c}(a', a_0) \geq v'(a) - \underline{c}(a, a_0) \geq v'(a) - p(a).$$

Consider now the incentives of a type $(v, B) \in f^{-1}(a)$ for which deviation to a' is affordable, i.e., $B \geq p(a')$. Because the price of a' does not change from its maximal bound $\overline{c}(a', a_0)$, at price $p(a)$ item a is now relatively more desirable:

$$v(a) - p(a) \geq v(a) - \overline{c}(a, a_0) \geq v(a') - \overline{c}(a', a_0) = v(a') - p(a'),$$

where the second inequality follows from the fact that maximal charges $\overline{c}(a, a_0), \overline{c}(a', a_0)$ implement f without deficits. □

5 Applications

In this section we present two application of our results. While we do not entertain yet a systematic search for revenue maximizing selling mechanisms, these applications point out to how our results can be used to accomplish such objectives, including in settings with multiple objects.

5.1 Multi-item Allocation with a Convex Type Space

Consider the following multi-item allocation problem. The seller has n different alternatives to allocate and must do so respecting some considerations given a priori. These external desiderata are expressed via the allocation function $f: \mathcal{T} \to \mathcal{A}$, where now $\mathcal{A} = \{a_1, \ldots, a_n\}$. The seller is allowed to maximize revenue given f, thus it searches for maximal prices that implement f without deficits. As a concrete example of this kind of situation, think of placements in a tertiary education system, where admission to different programs is merit-based, but fees take into account willingness and ability to pay.

The buyer demands at most one of the items the seller has to offer. In this section we assume that the buyer's valuation set is product space of open, bounded intervals; in particular, $\mathcal{V} = \times_{i=1}^{n} (\underline{v}_i, \overline{v}_i)$. We also assume that $\mathcal{B} \subset \mathbb{R}_+$ is an open, bounded interval. The type space in this subsection is $\mathcal{T} = \mathcal{V} \times \mathcal{B}$. We refer to it as a *convex type space*. We show that, under a mild assumption on the requirements imposed by f, maximal prices can be constructed as long as f satisfies our sufficiency condition for implementability. Moreover, we show that the derivation of maximal prices from the allocation network is straightforward. To state our results here more precisely, we need a bit more notation. First, perhaps after renaming, we index the set of alternatives $\mathcal{A} = \{a_1, \ldots, a_n\}$ according to the budget levels generated by

f; that is, let $\beta(a_1) \geq \beta(a_2) \geq \ldots \geq \beta(a_n)$. Second, in the allocation network $H = (N, E)$ generated by f, we define

$$\overline{\kappa}(a,a) \;=\; \underline{\kappa}(a,a) \;=\; 0 \qquad \text{for any } a \in N.$$

The following observation has been made in the context of dominant strategy implementation without budget constraints. But when using minimal capacities, it carries over our setting without modifications. This is where the structure of $\mathcal{V}$ as the product of open intervals plays a role. In what follows, it is important to recall that given the order imposed on alternatives, if $1 \leq i < j \leq n$, then $\overline{\kappa}(a_i, a_j) = \underline{\kappa}(a_i, a_j)$, as all types (v, B) in $f^{-1}(a_i)$ are able to afford a_j because of their larger budget set.

Lemma 2. *If the allocation network $H = (N, E)$ associated with f contains no cycle of negative minimal capacity, then for all $a_i, a_j \in \mathcal{A}$, the following holds:*

$$\underline{\kappa}(a_i, a_j) \;+\; \underline{\kappa}(a_j, a_i) \;=\; 0.$$

Proof. By assumption, all cycles containing edges in $E_1 = \mathcal{A} \times \mathcal{A}$ have non-negative minimal capacity. We show the validity of the claim for an arbitrary cycle $C = \{a_i, a_j, a_i\}$ in E_1. Suppose it were not true. Since minimal capacities in the cycle C are equal to the unrestricted incremental values, we obtain

$$\delta(a_i, a_j) \;+\; \delta(a_j, a_i) \;=\; \epsilon \;>\; 0.$$

For $0 < \epsilon' < \epsilon$, there exists a type $(v, B) \in f^{-1}(a_i)$ such that $\delta(a_i, a_j) \leq v(a_i) - v(a_j) < \delta(a_i, a_j) + \epsilon'$. For ϵ' sufficiently small, following arguments similar to Lavi et al. (2008) one can show that there exists a type $(\hat{v}, B) \in \mathcal{T}$ such that $\hat{v}(a_k) = v(a_k) - \epsilon'/2$ for all $k \neq j$, and $\hat{v}(a_j) = v(a_j) + \epsilon'/2$. Note that this implies

$$-\,\delta(a_j, a_i) \;<\; \hat{v}(a_i) \;-\; \hat{v}(a_j) \;<\; \delta(a_i, a_j).$$

Since $\hat{v}(a_i) - \hat{v}(a_k) = v(a_i) - v(a_k)$ for all $k \neq j$, one has that either $(\hat{v}, B) \in f^{-1}(a_i)$ or $(\hat{v}, B) \in f^{-1}(a_j)$. But either option yields to a contradiction to the definition of unrestricted incremental values. □

Lemma 3. *If the allocation network $H = (N, E)$ associated with f contains no cycle of negative minimal capacity, then for all $a_i, a_j, a_k \in \mathcal{A}$, one has that*

$$\underline{\kappa}(a_i, a_j) \;+\; \underline{\kappa}(a_j, a_k) \;+\; \underline{\kappa}(a_k, a_i) \;=\; 0.$$

Proof. By way of contradiction, suppose this is not the case, and let the cycle $C = \{a_i, a_j, a_k, a_i\}$ in $E_1 = \mathcal{A} \times \mathcal{A}$ have strictly positive minimal capacity. Then, immediately,

$$\begin{aligned} \underline{\kappa}(a_i, a_j) \;+\; \underline{\kappa}(a_j, a_k) \;+\; \underline{\kappa}(a_k, a_i) \;&>\; 0, \qquad \text{and} \\ \underline{\kappa}(a_i, a_k) \;+\; \underline{\kappa}(a_k, a_j) \;+\; \underline{\kappa}(a_j, a_i) \;&\geq\; 0. \end{aligned}$$

Adding these two inequalities obtains

$$\big(\underline{\kappa}(a_i, a_j) + \underline{\kappa}(a_j, a_i)\big) \;+\; \big(\underline{\kappa}(a_j, a_k) + \underline{\kappa}(a_k, a_j)\big) \;+\; \big(\underline{\kappa}(a_i, a_k) + \underline{\kappa}(a_k, a_i)\big) \;>\; 0.$$

In light of Lemma 2, this is impossible. □

An immediate consequence of Lemmas 2 and 3 is that for all alternatives a_i, a_j, a_k in $\mathcal{A}$,

$$\underline{\kappa}(a_i, a_j) + \underline{\kappa}(a_j, a_k) = \underline{\kappa}(a_i, a_k).$$

This implies that for any item $a_i \in \mathcal{A}$, the minimal charge $\underline{c}(a_i, a_0)$ is equal to either the minimal capacity of the direct edge (a_i, a_0), or the minimal capacity of a path $P = \{a_i, a_j, a_0\}$, for some $a_j \in \mathcal{A}$. Because by convention $\underline{\kappa}(a_i, a_i) = 0$, we can write

$$\underline{c}(a_i, a_0) = \min_{j=1,\dots,n} \left\{ \underline{\kappa}(a_i, a_j) + \underline{\kappa}(a_j, a_0) \right\}, \qquad \text{for all } a_i \in \mathcal{A}. \tag{3}$$

Lemma 4. *If the allocation network $H = (N, E)$ associated with f contains no cycle of negative minimal capacity, then there exists $a_{j^*} \in \mathcal{A}$ such that*

$$\underline{c}(a_i, a_0) = \underline{\kappa}(a_i, a_{j^*}) + \underline{\kappa}(a_{j^*}, a_0), \qquad \textit{for every alternative } a_i \in \mathcal{A}.$$

Proof. Suppose to obtain a contradiction that i^* is a solution to Equation 3 for a_i and k^* is a solution to Equation 3 for a_k, such that $i^* \neq k^*$. Assume without loss of generality that $\underline{\kappa}(a_i, a_{i^*}) + \underline{\kappa}(a_{i^*}, a_0) < \underline{\kappa}(a_i, a_{k^*}) + \underline{\kappa}(a_{k^*}, a_0)$. Adding $\underline{\kappa}(a_k, a_i)$ to both sides of this inequality, and using the insights from the previous lemmas obtains

$$\begin{aligned} \underline{\kappa}(a_k, a_{i^*}) + \underline{\kappa}(a_{i^*}, a_0) &= \underline{\kappa}(a_k, a_i) + \underline{\kappa}(a_i, a_{i^*}) + \underline{\kappa}(a_{i^*}, a_0) \\ &< \underline{\kappa}(a_k, a_i) + \underline{\kappa}(a_i, a_{k^*}) + \underline{\kappa}(a_{k^*}, a_0) = \underline{\kappa}(a_k, a_{k^*}) + \underline{\kappa}(a_{k^*}, a_0). \end{aligned}$$

This contradicts the fact that k^* was a solution to Equation 3. □

From Lemma 4 it follows that under a convex type domain the construction of implementable prices via minimal charges is straightforward. There is an alternative a_{j^*} that serves as a transit node between any a_i and the sink a_0 in the allocation network H. The price of a_{j^*} equals the budget level $\beta(a_{j^*})$. This construction does not state that prices are maximal. However, under a mild requirement on how f allocates alternatives, in the convex type domain case we find precisely this. Our assumption requires that, for any given alternative $a_i \neq a_1$, there is at least one type that can afford to buy item a_1 (and hence every other item too).

Proposition 9. *Let $\mathcal{T} = \mathcal{V} \times \mathcal{B}$ be a convex type space, and suppose that $f\colon \mathcal{T} \to \mathcal{A}$ is such that for all $i = 1, \dots, n$, there is a type $(v, B) \in f^{-1}(a_i)$ for which $B \geq \beta(a_1)$. If the allocation network H associated with f contains no cycle of negative minimal capacity, then the following holds:*

(a) Minimal and maximal capacities in the allocation network $H = (N, E)$ coincide; i.e., for all $a_i, a_j \in N$, one has that $\overline{\kappa}(a_i, a_j) = \underline{\kappa}(a_i, a_j) =: \kappa(a_i, a_j)$.

(b) There exists an alternative $a_{j^*} \in \mathcal{A}$ such that the maximal prices that implement f without deficits are given by

$$p^*(a_i) = \kappa(a_i, a_{j^*}) + \beta(a_{j^*}), \qquad \textit{for all } a_i \in \mathcal{A}. \tag{4}$$

Proof. (a) First observe that an implication of the property imposed on the allocation function f is that for all $a_i, a_j \in \mathcal{A}$, we have $\overline{\kappa}(a_i, a_j) < +\infty$. Let now $a_i, a_j \in \mathcal{A}$ be arbitrarily chosen.

Without loss of generality, assume that $i < j$. Because item a_j is affordable for all types in $f^{-1}(a_i)$, in this case we immediately have that $\overline{\kappa}(a_i, a_j) = \underline{\kappa}(a_i, a_j)$.

To show that $\overline{\kappa}(a_j, a_i) = \underline{\kappa}(a_j, a_i)$, notice first that we can repeat the arguments made in Lemma 2 to establish that

$$\overline{\kappa}(a_i, a_j) + \overline{\kappa}(a_j, a_i) = 0.$$

Indeed, recall that a_j is affordable for all types that are assigned a_i under f, and thus $\overline{\kappa}(a_i, a_j) = \delta(a_i, a_j)$. The previous equation implies that

$$\overline{\kappa}(a_j, a_i) = -\overline{\kappa}(a_i, a_j) = -\underline{\kappa}(a_i, a_j) = \underline{\kappa}(a_j, a_i),$$

where the last equality follows again from Lemma 2.

Thus, for any edge $(a_i, a_j) \in E_1 = \mathcal{A} \times \mathcal{A}$, we have that $\underline{\kappa}(a_i, a_j) = \overline{\kappa}(a_i, a_j) =: \kappa(a_i, a_j)$. Note that the equivalence between minimal and maximal capacities for edges in E_2 and E_3 follows from the definition of those capacities.

(b) Since all maximal and minimal capacities are the same in the allocation network $H = (N, E)$, Lemma 4 and Proposition 6 together show that the price function $p^* \colon \mathcal{A} \to \mathbb{R}$ defined by $p^*(a_i) = \kappa(a_i, a_{j^*}) + \beta(a_{j^*})$, for some $j^* \in \mathcal{A}$, is the maximal price function that implements f without deficits. $\square$

5.2 Revenue Maximization with a Discrete Type Space

Consider a setting where the seller possesses two items to allocate, a and a'. The exclusion option, which the seller can also execute, is denoted by a_0. The buyer has two possible private budgets, $\mathcal{B} = \{B_L, B_H\}$. We choose B_H so that types endowed with the high budget face no financial constraint; i.e., $B_H = 30$. The buyer has four possible valuations, $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$, where

	v_1	v_2	v_3	v_4
a	11	10	13	6
a'	16	20	28	26
a_0	0	0	0	0

Several situations give rise to these valuations. For example, a seller has two different land plots to allocate, one of them more valuable than the other, but the difference in values is dependent on the usage each type of buyer assigns to the plots. In Section 1 we provided the following interpretation: a seller has two units of the same good to allocate, say licenses to operate over a given airfreight route. Item a represents obtaining a single license, whereas a' represents obtaining both of them. Valuation differences are ordered to reflect an increasing degree of complementarities between licenses. A small carrier (v_1) has some capacity constraints, so the marginal value of obtaining two licenses instead of one is small. A large carrier (v_4), on the other hand, has high fixed costs and thus the value of operating a single route is relatively low. On the other hand, because of its size it can exploit strong complementarities in operating both licenses. Intermediate carriers (v_2 and v_3) lie between these two extremes.

Assume that each type $(v, B) \in \mathcal{V} \times \mathcal{B}$ occurs just once —i.e., types are uniformly distributed on $\mathcal{V} \times \mathcal{B}$— so there is no correlation between valuations and budgets. We consider revenue maximizing selling mechanisms under two different values for B_L while keeping the value for B_H fixed at 30.

Strong financial constraints Let $B_L = 5$, so that low budget types face strong financial constraints. Since a' is more valuable than a, for all valuations $v \in \mathcal{V}$, a simple way to separate types is by segmenting the market into low budget types, who are assigned a, and high budget types who get a'. Such an allocation function f specifies $f(v, B_L) = a$ and $f(v, B_H) = a'$, for all $v \in \mathcal{V}$. One readily obtains $\delta(a, a') = -20$, $\delta^r(a, a') = +\infty$, and $\delta(a', a) = \delta^r(a', a) = 5$, hence a necessary condition for implementability without deficits is $5 \geq p(a') - p(a) \geq -\infty$. From Proposition 8, we use the maximal charges in the allocation network to obtain maximal prices associated with f:

$$p(a) \;=\; \bar{c}(a, a_0) \;=\; 5 \qquad \text{and} \qquad p(a') \;=\; \bar{c}(a', a_0) \;=\; 10.$$

Expected revenue (recall each type occurs once) is then $R = 60$. The seller may decide to exclude low budget types, setting prices for a' equal to 16 and for a equal to 11; the price of the exclusion item a_0 is always zero. These new prices ensure that no buyer with a low budget participates in the market, but also that high budget types do not find it profitable to deviate by purchasing the low value item. In this case, revenue is $R = 64$, so full exclusion of low budget buyers is profitable for the seller.[17]

We show that the seller can do better by designing a selling mechanism that pools type (v_1, B_H) with the low budget types, instead of excluding them from the market. Consider an allocation function f^s that specifies

$$f^s(v_2, B_H) \;=\; f^s(v_3, B_H) \;=\; f^s(v_4, B_H) \;=\; a' \quad \text{and} \quad f^s(v, B) \;=\; a \quad \text{for all other types.}$$

The allocation graph associated to f^s is represented in Figure 4a, where the capacities of the edges (a, a_0) and (a', a_0) take into account participation constraints —thus, we are working with acceptable mechanisms here. The necessary condition for implementability without deficits is satisfied. Moreover, maximal prices are immediately derived from maximal chargers in the network:

$$p^s(a) \;=\; \bar{c}(a, a_0) \;=\; 5 \qquad \text{and} \qquad p^s(a') \;=\; \bar{c}(a', a_0) \;=\; 15.$$

The price of a is bounded, as before, by the low budget. However, by allocating a to type (v_1, B_H), the seller can increase the price differential between a and a', which increases profits. Types (v_3, B_H) and (v_4, B_H) buy a' and, at such prices, have no incentive to deviate by purchasing the low value item. On the other hand, type (v_2, B_H) is indifferent between purchasing a' and a, and hence follows the recommendation of the selling mechanism. Revenue is now $R = 70$. Revenue does not change if types (v_1, B_H) and (v_2, B_H) are both pooled with the low budget types.

Weak financial constraints Suppose now that $B_L = 8$, so that the financial constraints that low budget types face are weakened. One conjecture is that as financial restrictions ease, the

[17] One can readily verify from the respective allocation networks that no other form of exclusion is profitable for the seller.

optimal selling mechanism adjusts accordingly via prices. This conjecture is wrong because it does not take the incentive constraints into account. In particular, it is possible now that the price of the low value item be bounded by a valuation instead of B_L. In the current application, increasing the low budget changes the optimal allocation function to *exclude* type (v_4, B_L), creating efficiency loses to maximize expected revenue.

We start with market segmentation via income, so that low budget types purchase the low value item and high budget types buy the high value item, which yields expected revenue of $R = 68$ (the price for a is 6, the price for a' equals 11). Excluding all low budget types does not generate any gains for the seller. But, as in the case of strong budget constraints, the seller does increase revenue with a selling mechanism that assigns the low value item a to all low budget types and to type (v_1, B_H). The price of the low value item is not constrained by financial considerations but by $v_4(a)$, which acts now as the critical constraint. We obtain that the price for a is 6 and the price for a' is 16, for a total revenue of $R = 78$. The same expected revenue is obtained if the allocation function pools both (v_1, B_H) and (v_2, B_H) with the low budget types.

We claim now that the seller can still do better by excluding type (v_4, B_L) altogether. Indeed, consider the allocation function f^w that assigns

$$f^w(v_2, B_H) = f^w(v_3, B_H) = f^w(v_4, B_H) = a'$$

and

$$f^w(v_1, B_H) = f^w(v_1, B_L) = f^w(v_2, B_L) = f^w(v_3, B_L) = a.$$

Type (v_4, B_L) is excluded from the market and thus receives a_0.

The incremental values associated with f^w are $\delta(a', a) = \delta^r(a', a) = 10$, and $\delta(a, a') = -15 < -5 = \delta^r(a, a')$. This is represented in Figure 4b, which we can use to derive maximal prices that implement f^w without deficits:

$$p^w(a) = \bar{c}(a, a_0) = 8 \qquad \text{and} \qquad p^w(a') = \bar{c}(a', a_0) = 18.$$

The exclusion of (v_4, B_L) does not affect the price differential between the high and the low value item. However, it allows the seller to increase prices for both items. All low budget types who purchase a are better off than if they choose the exclusion option a_0. At these prices, a buyer with type (v_2, B_H) is indifferent between purchasing the low and the high value item. Finally types (v_3, B_H) and (v_4, B_H) obtain a positive rent from the purchase of a' and have no incentive to purchase the low value item. Expected revenue derived from this selling mechanism is $R = 86$. Revenue does not change if the seller allocates item a' to (v_3, B_H) and (v_4, B_H), and a to all other types except (v_4, B_L) who remains priced out of the market. The seller won't benefit from excluding any other type of buyer. Thus $\{f^w, p^w\}$ is the optimal selling mechanism under weak financial constraints.

Correlation of budgets and valuations We stress that because in our approach the derivation of prices comes directly from the allocation network, it allows us to accommodate correlation of budgets and valuations in a straightforward way. Indeed, for each allocation function that is implementable without deficits we obtain maximal prices in the 2-item case. Because maximal prices are maximal charges in the allocation network, they do not depend on the frequency of types but rather on the maximal capacities. Thus, expected revenue can simply be computed using the distribution of types.

In the current application with weak financial constraints, using the interpretation of v_1 and v_4 being small and large buyers, a positive correlation of size and financial constraints (so that types (v_1, B_L) and (v_4, B_H) occur more often than all other types) has no effect on the optimal selling mechanism. On the other hand, whether or not a negative correlation of size and budgets alters the optimal mechanism depends on the degree of correlation. As the reader can easily verify, if types (v_1, B_H) and (v_4, B_L) are four times as likely as all other types to occur, then the optimal selling mechanism under weak financial constraints is no longer f^w but f^s, so that no type is optimally excluded from the market.

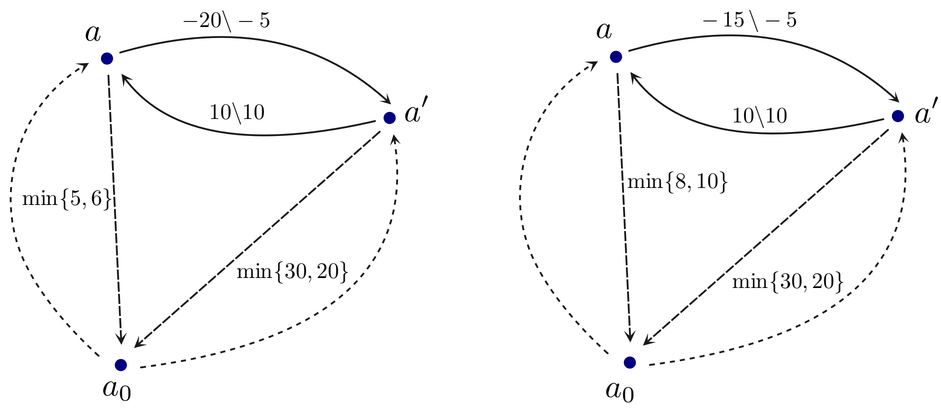

Figure (a) Figure (b)

Figure 4: Allocation networks for optimal selling mechanisms with (a) strong and (b) weak budget constraints.

6 Concluding Remarks

This paper has introduced a novel network flow approach to studying selling mechanisms that are implementable without deficits in the presence of financial constraints from the part of the buyer. Our framework is quite general and considers allocation problems with multiple units or multiple objects to sell. Our results exploit the subtle difference between incremental values and restricted incremental values between any two alternatives. We have argued that these concepts, which depend on the allocation function that is being considered, encode all the relevant information required by the seller to construct implementable prices that respect the budget constraints of the buyer and do not raise a deficit for the seller. We have shown via two applications how this construction can be accomplished directly from the allocation network associated to a given allocation function.

Because our general results do not rely on any specific structure of the design problem, we believe they will be useful to advance current available methods for addressing related problems of mechanism or market design with financial constraints, e.g., revenue maximization, efficiency, etc. We have not considered dynamic selling mechanisms, or mechanisms where any proof of financial or liquidity positions are available to the buyer (and demanded by the seller). Neither have we entertained the possibility of the seller offering subsidies to some types of buyers, so we don't know if, or when, such possibility will increase expected revenue for the seller. We leave these interesting topics for future research.

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