Constrained Random Matching

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Abstract

This paper generalizes the Probabilistic Serial (PS) mechanism of Bogomolnaia and Moulin (2001) to matching markets with arbitrary constraints. The constraints are modeled as a list of permissible ex-post allocations. The method described here computes simple linear inequalities such that when a general version of the PS algorithm is executed under these inequalities, the outcome is an sd-efficient lottery over the permissible set of allocations. The inequalities correspond to the hyperplanes defining a convex polytope that is intuitively constructed from the given constraint set. The method is general, can be applied to both one-sided and two-sided matching markets, and allows for multi-unit demand.

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*L4 111 Barry St, Department of Economics, The University of Melbourne VIC 3010, Australia. Email: ivan.balbuzanov@unimelb.edu.au. Website: https://sites.google.com/site/ibalbuzanov/ This paper is based on Chapter 2 of my PhD dissertation and was previously circulated as the second half of a longer manuscript titled "Short Trading Cycles: Kidney Exchange with Strict Ordinal Preferences." I am grateful to my dissertation advisor, Haluk Ergin, for his encouragement, support, and guidance. This paper has benefited from comments and suggestions by David Ahn, Eric Auerbach, Haris Aziz, Aaron Bodoh-Creed, Yeon-Koo Che, Satoshi Fukuda, Yuichiro Kamada, Fuhito Kojima, Maciej Kotowski, C. Matthew Leister, Simon Loertscher, John Mondragon, Michèle Müller-Itten, Takeshi Murooka, Paulo Natenzon, Omar Nayeem, Aniko Öry, Chris Shannon, Emilia Tjernström, M. Utku Ünver, Tom Wilkening, Steven Williams, and a number of seminar audiences. The research was partly supported by NSF grant SES-1227707, which I gratefully acknowledge.

1 Introduction

Lotteries are widely used in object-allocation settings whenever monetary transfers are absent. Examples include the assignment of public housing (Thakral, forthcoming), of school seats or university courses to students (Pathak 2017), of players' contract rights to sports teams via the draft, or of military and jury duty to citizens. The practical popularity of lotteries is a consequence of the fact that efficiency and fairness can typically be jointly maintained only by using randomization (Abdulkadiroğlu and Sönmez 1998).

Outcomes in these environments often have to conform to binding constraints. For example, the location preferences of public-housing applicants make certain addresses unacceptable while a public-housing authority might want to maximize the utilization of the managed housing stock and the number of families served while maintaining a minimum level of diversity. In addition to school capacity, school-choice problems can be subject to quotas and caps coming from demand for ethnic, gender or socio-economic representation or the need to respect students' priority rankings at different schools. Constraints, however, limit which outcomes are possible and, thus, present a challenge to implementing fair lotteries in practice.

This paper will take a dual approach to constraints by focusing instead on the set of all ex-post allocations satisfying the given constraints. For example, the set of all permissible (i.e. constraints-satisfying) public-housing allocations may be the set of all allocations that are individually rational for the participants, satisfy a diversity criterion, and meet a certain threshold of utilization. Thus, any allocation lottery must assign positive probability only to permissible allocations in order to satisfy the constraints.

Motivated by this dual representation of constraints, I consider the problem of random object-assignment under the simple requirement that an assignment lottery places positive probability only on allocations in a given (arbitrary) set. I propose a method of satisfying this requirement for any such list or permissible allocations. The mechanism I propose, the *Generalized Constrained Probabilistic Serial* (GCPS) mechanism, accommodates arbitrary constraints while maintaining efficiency and anonymity.

To be more precise, assume that a list of permissible ex-post allocations is given so that a random matching can put positive probability only on allocations on the list. Given such a set of *outcome constraints*, I propose a simple way of computing a set of linear inequalities (the *algorithm inequalities*) which, when used as constraints during the run of a simultaneous greedy-eating algorithm, guarantee that the output represents a random allocation satisfying two desirable properties. First, it is a lottery over allocations satisfying the outcome constraints (i.e. a lottery whose support is a subset of the permissible set).

Second, it is sd-efficient (i.e. Pareto optimal with respect to first-order stochastic dominance, an appropriate efficiency notion for random environments) among all such lotteries over permissible allocations. It is important to note that, as we will see below, even if the permissible matchings can be characterized by a set of inequalities, these inequalities cannot generally be used as algorithm inequalities.

The results apply to a broad range of settings: object-allocation and object-exchange problems as well as two-sided matching markets with single- or multi-unit demand, such as the college-admission problem or the problem of assigning medical interns to hospitals. The permissible set can be any non-empty set of deterministic allocations, including those derived from constraints such as individual rationality (Example 2), maximum caps (Example 3), minimum quotas (Example 4), maximum length of exchange cycles in object-exchange problems (Example 6), or any combination of these.

Of course, the GCPS is a general version of the seminal and widely studied Probabilistic Serial (PS) mechanism (Bogomolnaia and Moulin 2001; BM henceforth). It is instructive to consider the difficulty one faces when trying to adapt the PS mechanism to a constrained environment. The way the PS algorithm works in the simple unconstrained setting considered by BM, where n indivisible and heterogeneous objects need to be assigned to n agents, is as follows. After agents submit preferences, the mechanism treats each object as if it is perfectly divisible and each agent as if she is continuously (and contemporaneously with all other agents) claiming (or "eating") ever-increasing shares from her most preferred available object with some exogenously given speed. Once the unit mass of an object has been completely claimed, each agent currently eating it moves to her next most preferred available object. Once the sum of shares claimed from all objects equals each agent's demand, the algorithm ends. The claimed shares form a $n \times n$ bistochastic matrix (i.e. a non-negative matrix with each row and column summing up to 1) and are treated as probability shares: i.e. an object o is assigned to agent a with probability equal to the share a claimed from o or, equivalently, the (a, o) entry of the matrix. The Birkhoff-von Neumann theorem guarantees that the bistochastic matrix represents a random allocation that is a lottery over deterministic agent-object allocations.

The PS mechanism easily deals with the objects' capacity constraints (each object comes in a single copy) as well as the analogous constraints coming from each agent's unit demand: BM treat outcome constraints by directly using them as algorithm inequalities. Even though the PS algorithm performs well under this approach, this does not hold for more general constraints. In fact, applying just the inequalities characterizing the set of allowable deterministic matchings to the higher-dimensional set of interim probability-share matrices arising during the run of a PS-type algorithm is in general insufficient to guarantee that the algorithm's output satisfies the two conditions above. Thus, it is important to differentiate between the outcome constraints and the algorithm inequalities.

As an illustration, consider the problem of guaranteeing ex-post individual rationality (Yılmaz 2010). Deterministic allocations satisfying individual rationality are characterized by the inequality $m(i, o) \leq 0$ for any object o that is not individually rational for agent i, where $m(i, o) \in \mathbb{Z}_+$ is the number of copies of object o that agent i receives. However, attempting to run the PS algorithm by using the inequalities characterizing individual rationality as algorithm inequalities does not work. As an example, assume that there are two agents, a_1 and a_2, and two objects, o_1 and o_2. Only object o_1 is individually rational for agent a_1, while both objects are individually rational for agent a_2 but she prefers o_1. A simple constraint fully characterizing the set of individually rational random matchings is $m(1, 2) \leq 0$. If the PS algorithm is run with this as an added inequality, however, it would *not* guarantee individual rationality. To see that, observe that if a_2 starts claiming probability shares from her favorite object o_1, this would make the only individually rational allocation, in which a_1 receive o_1 for sure, impossible.[1]

Other settings in which the application of the PS mechanism using the outcome constraints as algorithm inequalities leads to a failure include minimum-quota constraints. For example, consider the requirement that at least one copy of a certain object o_1 is assigned to either agents a_1 or a_2 or, formally, $m(1, 1) + m(2, 1) \geq 1$. It is unclear how one should adapt this constraint to the PS algorithm. The matrix used to keep track of the claimed probability shares is initialized to zero and if o_1 is not high in the preference order of either a_1 or a_2, it is not clear how they can be induced to claim shares from that object.[2] The approach developed in this paper surmounts this issue: see Example 4 for a specific application.

A constraint requiring that agent a_1 receives weakly more copies of object o_1 than a_2 does leads to similar issues. First, note that if a random matching satisfies the corresponding linear inequality $m(1, 1) \geq m(2, 1)$ this can only guarantee that the desired condition is satisfied *in expectation*. Moreover, even this weaker condition cannot be satisfied by a direct application of a PS-type algorithm. If o_1 is a_2's favorite object, a_2 would claim probability shares from o_1 at the start of the algorithm. It is again unclear how the mechanism can induce a_1 to claim shares from o_1 without jeopardizing efficiency, especially if that object is low in a_1's preferences. See Example 5 for how this paper's approach can overcome this issue.

The general approach provided here is applicable also when the outcome constraints are not readily expressed as inequalities. As an example, consider the application of the PS

[1] To get around this difficulty, Yılmaz (2010) finds algorithm inequalities guaranteeing individual rationality via an ingenious application of Gale's Supply-Demand Theorem (Gale 1957). See also Example 2.

[2] Budish et al. (2013) note that a limitation of their Generalized Probabilistic Serial mechanism is its inability to deal with minimum quotas.

mechanism to the problem of pairwise kidney exchange (Roth et al. 2005a). The outcome of the algorithm must be a random allocation that can be implemented as a lottery over deterministic matchings, each one of which represents a set of exchanges, none of which involves more than two patient-donor pairs. There is no obvious system of inequalities characterizing the set of random allocations satisfying these outcome constraints. Balbuzanov (2018) uses Edmonds' characterization of the matching polytope (Edmonds 1965) to derive such a system. See also Example 6.

The algorithm inequalities of Yılmaz (2010) and Balbuzanov (2018), as well as BM's bistochastic constraints and Budish et al.'s (2013) more general *bihierarchical constraints* (which are both outcome constraints and algorithm inequalities in their settings), all take the same inequality form: sums of probability shares (possibly scaled by positive scalars) not exceeding a non-negative number. This paper refers to these inequalities as *positive linear inequalities*. As the interim probability-share matrices of the PS algorithm are increasing in the time component, other types of algorithm inequalities might jeopardize the sd-efficiency of the algorithm's output. Since agents claim object shares in the order of their preferences, if, for example, an algorithm inequality becomes slack after binding, the resulting outcome might not be sd-efficient. So, intuitively, in order to preserve sd-efficiency, the algorithm inequalities must be of a form where, once binding, they remain binding throughout.

Until now, it has been an open question whether a set of well behaved positive linear inequalities exists for each possible configuration of outcome constraints (i.e. all possible lists of permissible deterministic matchings). One has to walk a fine line in the derivation of these inequalities. On the one hand, it is easy to guarantee that a mechanism selects an allowable allocation by choosing overly strict algorithm inequalities. The designer can merely choose one of the permissible matchings m and force it by allowing agents to only claim shares from the object that they are assigned under m, regardless of their preferences. This, of course, may not be efficient. On the other hand, if the algorithm inequalities are too loose, a PS-type algorithm may fail to output a permissible random allocation as shown above.

Proposition 1 answers this question in the affirmative. Furthermore, the procedure implied by Proposition 1 not only guarantees that the outcome of the algorithm satisfies the outcome constraints but also that it would be constrained efficient with respect to the agents' preferences (Proposition 2). In other words, the algorithm inequalities implied by Proposition 1 are weak without being too weak: they do not rule out any possible allowable allocations while simultaneously guaranteeing that the final outcome of the PS algorithm, when subject to these inequalities, represents a lottery over permissible allocations.

Mathematically, the paper relies on the observation that, as the interim probability-share matrices increase in the algorithm's time variable, any interim matrix M^t occurring at time

t must satisfy $M^t \leq M$ for some $M \in \Delta C$, where ΔC is the convex hull of the list of permissible deterministic matchings. Thus, the algorithm inequalities must not allow an interim matrix to "leave" the *lower contour set* of ΔC, defined by

$$\{M \in \mathbb{R}_+^{n \times k} | \exists M' \in \Delta C : M \leq M'\}.$$

The proof of Proposition 1 establishes that this set is a bounded convex polytope and can be defined as all points in $\mathbb{R}_+^{n \times k}$ satisfying a finite set of *positive* linear inequalities. Proposition 2 then shows that if these inequalities are used as the algorithm inequalities bounding the matrices arising during the run of the PS mechanism, the resulting outcome is (constrained) sd-efficient.

1.1 Literature Review

The GCPS mechanism extends Bogomolnaia and Moulin's (2001) Probabilistic Serial mechanism, initially defined for the object-assignment setting with strict preferences and single-unit demand. Since their seminal contribution, their work has been generalized for ordinal preferences allowing indifferences (Katta and Sethuraman 2006), for multi-unit demand (Kojima 2009), for property rights necessitating individual rationality (Yılmaz 2009, 2010), for fractional endowments (Athanassoglou and Sethuraman 2011), and for combinatorial demand (Nguyen et al. 2016).[3]

Some important recent work on general random matching mechanisms include Budish (2011), Budish et al. (2013), Kesten and Ünver (2015), Pycia and Ünver (2015), and Akbarpour and Nikzad (2017). Budish et al. (2013) is the closest to my paper. One of its contributions is providing a sufficient condition for constraints (comprised of lower- and upper-bound quotas) to satisfy the following appealing property called *universal implementability*: whenever a bistochastic matrix satisfies the constraints, it can be decomposed as a convex combination of permutation matrices, such that each one of them satisfies the same constraints (regardless of the desired lower and upper bound of each constraint). The present paper does not rely on universal implementability and, as emphasized above, in a further departure from Budish et al. (2013), decouples the outcome constraints from the algorithm inequalities. I examine Budish et al.'s (2013) Generalized Probabilistic Serial mechanism as a special case of GCPS in Example 3. Budish (2011) and Akbarpour and Nikzad (2017)

[3]Other work spurred on by Bogomolnaia and Moulin (2001) include characterizations of sd-efficiency (McLennan 2002, Abdulkadiroğlu and Sönmez 2003, Manea 2008, Carroll 2010), the study of the behavior of the PS mechanism in large markets (Che and Kojima 2010, Kojima and Manea 2010, Liu and Pycia 2016), and axiomatic characterizations of the PS mechanism and its extensions (Bogomolnaia and Heo 2012, Heo and Yılmaz 2015, Hashimoto et al. 2014, Heo 2014a,b).

who propose random mechanisms satisfying desirable properties approximately, while Pycia and Ünver (2015) study which random mechanisms are decomposable into deterministic mechanisms with the same properties.

The literature around accommodating, relaxing or redefining stability in (deterministic) two-sided markets with constraints have grown substantially in recent years: e.g. Hafalir et al. (2013), Ehlers et al. (2014), Kamada and Kojima (2015, 2017). For more references, see Kojima (forthcoming) for a recent short informal survey. I address the question of stability in Example 7.

2 Model

Let $A = \{1, \dots, n\}$ be a set of agents and $O = \{o_1, \dots, o_k\}$ a set of objects. Each object has $q_{o_i} \in \mathbb{N}$ copies and each agent demands a total of $d \in \mathbb{N}$ object copies for consumption (with possibly multiple copies from the same object). Without loss of generality, assume that $\sum_{i=1}^{k} q_{o_i} \geq nd$.[4] Each $i \in A$ has an associated strict preference order $\succ_i$ over O. Denote all possible strict preference orders over O by $\mathbb{P}$ and all possible profiles for the n agents by $\mathcal{P} := \mathbb{P}^n$ with generic element $\succ$. A **deterministic matching** of agents to objects is a matrix m which satisfies $m(i, j) \in \{0, 1, \dots, d\}$ for all (i, j) and

$$\begin{aligned} &\sum_{j=1}^{j=k} m(i, j) = d \text{ for all } i \in \{1, 2, \dots, n\}; \\ &\sum_{i=1}^{i=n} m(i, j) \leq q_{o_j} \text{ for all } j \in \{1, 2, \dots, k\}. \end{aligned} \tag{1}$$

We interpret the value $m(i, j)$ as indicating that agent i receives $m(i, j)$ copies of object o_j. System (1) implies that each agent receives exactly d object copies in total, and no object is assigned in excess of its supply. The set of all deterministic matchings is denoted by $\mathcal{M}$.

Let $\Delta\mathcal{M}$, the convex hull of $\mathcal{M}$, be the set of **random matchings**. Each random matching $\mu \in \Delta\mathcal{M}$ is a convex combination of some subset of elements from $\mathcal{M}$. Thus $\mu(i, j)$ denotes the expected number of copies of object o_j that agent i receives and row i of μ denotes the expected number of copies agent i receives from each of the k objects. Call this agent i's **probability-share allocation** and denote it by $\mu(i)$. Note that μ satisfies (1). The converse holds as well: by Theorem 1 of Budish et al. (2013), if a non-negative matrix μ satisfies (1), it is expressible as a convex combination of elements of $\mathcal{M}$ and therefore $\mu \in \Delta\mathcal{M}$. A **sub-**

[4] In particular, O can include a fictitious null object that is assigned to agents if and only if they receive fewer than d copies from all (real) objects.

random matrix M is any non-negative matrix which satisfies the following relaxed version of (1):

$$\sum_{j=1}^{j=k} M(i,j) \le d \text{ for all } i \in \{1,2,\dots,n\};$$
$$\sum_{i=1}^{i=n} M(i,j) \le q_{o_j} \text{ for all } j \in \{1,2,\dots,k\}.$$

Assume that $\succ_i$ ranks the objects in O in the order $(o_1, o_2, \dots, o_k)$ from best to worst. Consider p and q to be two probability-share allocations and let p_{o_j} and q_{o_j} denote the expected number of copies that agent i receives from object o_j. Then p **first-order stochastically dominates** q **with respect to** $\succ_i$ if

$$\sum_{l=1}^{j} p_{o_l} \ge \sum_{l=1}^{j} q_{o_l} \tag{2}$$

for each $j = 1, \dots, k$. Further, p **strictly first-order stochastically dominates** q **with respect to** $\succ_i$ if (2) holds and $p \neq q$.

A function $f : \mathcal{P} \to \Delta\mathcal{M}$ is a **random mechanism**. A random mechanism f is **sd-efficient** if for all $\succ$ there does not exist an element $\mu \in \Delta\mathcal{M}$ such that $\mu(i)$ first-order stochastically dominates $f(\succ)(i)$ with respect to $\succ_i$ for all $i \in A$ and strictly so for some $i \in A$.

A correspondence $C : \mathcal{P} \to 2^{\mathcal{M}} \setminus \{\varnothing\}$ is a **constraint correspondence**: $C(\succ)$ is the set of *allowable* (or *permissible*) ex-post deterministic matchings for the preference profile $\succ$ or, equivalently, the matchings satisfying the *outcome constraints*. The set $\Delta C(\succ)$ contains all random matchings that are *implementable* as a convex combination of allowable deterministic matchings.

The property of anonymity (or name-invariance) applies to both constraints and mechanisms. Given a bijection $\pi : A \to A$ and some $m \in \mathcal{M}$, let $\Pi^\pi(m)$ be defined from m via $\Pi^\pi(m)(i,j) = m(\pi^{-1}(i), j)$. Assuming that $\succ_i$ for some $i \in A$ ranks the objects from O in the order $(o_1, \dots, o_k)$, let $\succ^\pi_{\pi(i)}$ be the preference relation corresponding to the same order $(o_1, \dots, o_k)$. Then a constraint correspondence C is **anonymous** if for all $\succ \in \mathcal{P}$ and all π one has $\Pi^\pi(C(\succ)) = C(\succ^\pi)$. A random mechanism f is **anonymous** if for all $\succ$ the matrices $f(\succ)$ and $f(\succ^\pi)$ are identical up to the permutation π. In other words, $f(\succ)(i,j) = f(\succ^\pi)(\pi(i), j)$ for all i, j.[5]

[5] If A and O are not disjoint as is the case in some of the examples below, the elements in O need to be appropriately permuted as well. Namely, if $A = O$, the preference relation $\succ^\pi_{\pi(i)}$ is $\pi(o_1) \succ^\pi_{\pi(i)} \cdots \succ^\pi_{\pi(i)} \pi(o_k)$,

Given a set $C' \subset \mathcal{M}$ and some $\succ \in \mathcal{P}$, a random matching $\mu \in \Delta C'$ is C'**-constrained sd-efficient with respect to** $\succ$ if there does not exist a $\mu' \in \Delta C'$ such that $\mu'(i)$ first-order stochastically dominates $\mu(i)$ with respect to $\succ_i$ for all $i \in A$ and strictly so for some $i \in A$. Similarly, a deterministic matching $m \in C'$ is C'**-constrained efficient with respect to** $\succ$ if it is C'-constrained sd-efficient with respect to $\succ$. Given a constraint correspondence C, a mechanism $f : \mathcal{P} \to \Delta\mathcal{M}$ is C**-constrained sd-efficient** if every $f(\succ)$ is $C(\succ)$-constrained sd-efficient with respect to $\succ$.

3 The GCPS and Its Properties

As discussed in the introduction, in order to adapt the PS mechanism to a constrained setting, one needs to derive appropriate linear inequalities and apply them to the sub-random matrices arising during the run of the algorithm. These inequalities need to be derived from the set of permissible allocations in a way that guarantees that the PS mechanism selects an implementable lottery. Towards a formalization, let Ω^0 be the collection of all pairs (a, b) comprised of a function $a : A \times O \to \mathbb{R}_+$ and a scalar $b \in \mathbb{R}_+$. Each one of these pairs determines a *positive linear inequality*:

$$\sum_{(i,j) \in A \times O} a(i,j) M(i,j) \leq b.$$

Note that Ω^0 contains "non-positivity" inequalities, such as $M(i,j) \leq 0$, representing the case $M(i,j) = 0$ for all interim matrices M (as such matrices are weakly positive). Other types of inequalities, such as, for example, $M(i,j) - M(h,l) \leq 1/2$ or $M(i,j) + M(h,l) \geq 1$, are not represented by elements of Ω^0 as they would present substantial difficulties for PS-type algorithms. The first one of these has the potential to become slack after binding during the course of the algorithm. This is problematic since agent i would have stopped claiming shares from object o_j and moved on to her next-best available object. If that inequality becomes slack again, however, agent i could benefit from coming back to object o_j and claiming more shares from it, which PS-type algorithms do not allow. This would jeopardize the efficiency of the outcome. The second of these inequalities is not initially satisfied for the initial matrix $M^0 = \mathbf{0}$ and, therefore, there is no guarantee that a PS-type algorithm would lead to a final outcome satisfying that inequality.[6]

and f is anonymous if $f(\succ)(i,j) = f(\succ^{\pi})(\pi(i), \pi(j))$.

[6] It is very important to highlight the fact that even though Ω^0 does not include these inequalities, the treatment of constrained random matching in this paper allows the corresponding outcome constraints to be honored. That is to say, even though $M(i,j) - M(h,l) \leq 1/2$ and $M(i,j) + M(h,l) \geq 1$ do not correspond to members of Ω^0, they could be part of the constraints defining the set $C(\succ)$. See, for instance, Example 4.

Once the set of positive linear inequalities $\Omega \subset \Omega^0$ corresponding to some outcome-constraint set $C(\succ)$ are found (more on that below), the generalized constrained PS mechanism subject to Ω is defined as follows.

Definition 1. Generalized Constrained Simultaneous-Eating and Probabilistic Serial mechanisms subject to Ω. Each $i \in A$ has an associated claiming-speed function $e_i : [0,1] \to \mathbb{R}_+$ with $\int_0^1 e_i(t)dt = d$. Time runs continuously starting at $t = 0$. For each point in time there is an associated sub-random matrix M^t where M^0 is the initial zero matrix. Object o_j is *available* to agent i at time t if none of the inequalities in Ω for which $a(i,j) > 0$ binds at that time. Note that M^0 satisfies all the inequalities in Ω. At time t, each agent i claims with speed $e_i(t)$ the remaining probability shares of her favorite reported object o_j among the objects that are available to her at that instance.[7] That increases the probability that i receives object o_j—i.e. it increases $M^t(i,j)$. The algorithm ends at time $t = 1$ and its output is the random matching M^1.

Denote the outcome of the *generalized constrained simultaneous-eating (GCSE) algorithm subject to* Ω given a preference profile $\succ$ by $GCSE(\succ, e, \Omega)$. If the claiming-speed functions for all agents are the same (assumed, without loss of generality, to be $e_i(t) = d$ for all $i \in A$ and $t \in [0,1]$), call the resulting mechanism the *generalized constrained probabilistic serial (GCPS) mechanism subject to* Ω and denote its outcome by $GCPS(\succ, \Omega)$.

Let the set of allowable matchings satisfying the outcome constraints be C'. If at any time during the algorithm's run, there is a sub-random matrix M^t such that there does not exist $M' \in \Delta C'$ with $M' \geq M^t$, the algorithm would not output an implementable random matching. This follows from the fact that the interim sub-random matrices are increasing in t. Thus, at the very least, we need the inequalities Ω to guarantee that for all t there is some $M' \in \Delta C'$ such that $M' \geq M^t$. In other words, if we define the *lower contour set* of a set $X \subset \mathbb{R}_+^{n\times k}$ to be

$$\text{lcs}(X) := \{M \in \mathbb{R}_+^{n\times k} | \exists M' \in X : M \leq M'\},$$

we require that all sub-random matrices stay within the lower contour set of $\Delta C'$. Proposition 1 shows that any lower contour set of a subset of $\mathcal{M}$ can be fully characterized by a set of positive linear inequalities.

Proposition 1. *For any set $C' \subset \mathcal{M}$, there exists an essentially unique minimal set of*

[7] To see why the set of available objects cannot be empty for any i and any $t < 1$, see the proof of Proposition 2.

inequalities $\Omega^{C'} \subset \Omega^0$ *such that*

$$\text{lcs}(\Delta C') = \bigcap_{(a,b)\in\Omega^{C'}} \left\{ M \in \mathbb{R}_+^{n\times k} \,\middle|\, \sum_{(i,j)\in A\times O} a(i,j)M(i,j) \leq b \right\}.$$

Proposition 1 indicates that there exists a set of positive linear inequalities any sub-random matrix needs to satisfy in order to belong the lower contour set of $\Delta C(\succ)$ and in order for the GCSE mechanism to output an implementable random matching.[8] Proposition 2 secures the sufficiency: constraining the sub-random matrices to the lower contour set of $\Delta C(\succ)$ is also sufficient in that the algorithm not only terminates at an implementable random matching but also that matching is also constrained sd-efficient.

Proposition 2.

i. For any $\succ\in\mathcal{P}$*, any non-empty set* $C(\succ)\subset\mathcal{M}$*, and any claiming-speed profile* e*,* $GCSE(\succ, e, \Omega^{C(\succ)})$ *is a random matching from* $\Delta C(\succ)$ *that is* $C(\succ)$*-constrained sd-efficient with respect to* $\succ$*.*

ii. If C *is an anonymous constraint correspondence, the GCPS mechanism subject to* $\Omega^{C(\succ)}$ *is anonymous in addition to being* C*-constrained sd-efficient.*

Proposition 2 states that, regardless of the list of allowed deterministic matchings or agents' preferences, the GCSE mechanism subject to $\Omega^{C(\succ)}$ always selects a constrained sd-efficient random matching.[9] A fortiori, GCPS subject to $\Omega^{C(\succ)}$ is constrained sd-efficient. Furthermore, it is anonymous if the constraint correspondence allows name-invariant treatment of agents (i.e. if it is anonymous itself).

The main mathematical observation behind Propositions 1 and 2 is that any lower contour set is a bounded convex polytope and so it is the set of all matrices whose entries

[8]The set of characteristic inequalities is also "essentially" unique in the sense that for the pairs (a, b) with $b > 0$, the inequalities are unique up to rescaling by a positive scalar. Multiple sets of inequalities can be equivalent to each other whenever $b = 0$, however: for example, the inequalities $M(1,1) \leq 0$ and $M(2,1) \leq 0$ are jointly equivalent to $M(1,1)+M(2,1) \leq 0$ and to $M(1,1)+2M(2,1) \leq 0$. They all denote the restriction that $M(1,1) = M(2,1) = 0$ for all allowable allocations. The uniqueness of the set of inequalities is not used in the sequel. The existence of a set of *positive* linear inequalities characterizing the lower contour set is the crucial take-away of Proposition 1.

[9]Such a matching μ is only decomposable as a convex combination of constrained efficient deterministic matchings (otherwise we can improve on μ in first-order stochastic dominance sense by shifting some of the probability weight from a constrained inefficient to a constrained efficient deterministic matching; see BM for a more detailed argument). Therefore, instead of working with the full list of allowed ex-post matchings $C(\succ)$, it is sufficient to know the set $PO(C(\succ))$ of constrained efficient matchings in $C(\succ)$ with respect to $\succ$. The GCSE mechanism subject to the inequalities $\Omega^{PO(C(\succ))}$ would terminate at the same matching as the GCSE mechanism subject to the inequalities $\Omega^{C(\succ)}$. This alternative approach could be beneficial if it improves the time complexity of the computation of the inequalities defining the lower contour set. Of course, this also depends on the relative complexity of computing $PO(C(\succ))$ given $C(\succ)$.

satisfy a finite set of inequalities and, importantly, those inequalities are all positive linear inequalities.[10] In other words, those inequalities are from the set Ω^0.[11] As noted above, once a positive linear inequality starts binding during the run of the GCSE algorithm, it remains binding until the algorithm's conclusion. Thus, whenever an inequality binds and an object becomes unavailable to an agent, that agent can be moved to claiming shares from her next highest-ranked available object and the designer need not worry about returning the agent to the object whose probability shares she was just claiming in case the inequality ever becomes slack again. Thus sd-efficiency is not jeopardized.

4 Examples

The approach described above can accommodate any set of outcome constraints after boiling them down to a list of allowable deterministic matchings. This section presents a handful of examples illustrating not just the power of this approach but also how to apply it in different settings. Examples 1, 2, and 3, for instance, show not only that the sd-efficiency results of a handful of preceding papers are implied by Proposition 2 but also that the inequalities that they use to guarantee implementability are special cases of the inequalities defining the lower contour set of the convex hull of all permissible matchings.

Example 1. (Unconstrained object assignment.)

If $n = k$, $d = q_o = 1$ for all $o \in O$, and $C(\succ) = \mathcal{M}$ for all $\succ \in \mathcal{P}$, the model above reduces to the unconstrained pure object-assignment model of BM. Each deterministic matching is a permutation matrix (i.e. a matrix that has exactly one entry equal to 1 in each row and in each column with all other entries zero), and each random matching is a bistochastic matrix (i.e. the sum of its entries along any given row or column is 1). The constant constraint correspondence permits randomization over any possible deterministic matching. The inequalities defining $\text{lcs}(\Delta\mathcal{M})$ are the sub-bistochasticity conditions: $M \in \text{lcs}(\Delta\mathcal{M})$ if and only if

$$\sum_j M(i,j) \leq 1 \text{ and } \sum_i M(i,j) \leq 1 \text{ for all } i \in A, j \in \{1,\dots,k\}. \tag{3}$$

[10] These exclude the non-negativity constraints but those are automatically satisfied since the interim sub-random matrix is initialized to the zero matrix and it is increasing in t.

[11] Computing these inequalities given the set of extreme points is known as the *facet enumeration problem*, a particular case of the *convex hull problem*. This problem has been extensively studied in the computational-geometry literature (Motzkin et al. 1953, Chand and Kapur 1970, Avis and Fukuda 1992, Chazelle 1993, Seidel 2018). For practical purposes, the examples in this paper were computed using `cdd` (Fukuda 1999), verifying the computations with `lrs` (Avis 2000) and/or Qhull (Barber et al. 1996) whenever feasible. See Appendix B for more details on the computational aspects of this problem.

These are the algorithm inequalities used by BM's PS mechanism.

Example 2. (Individual rationality.) Maintaining the assumptions of Example 1 with the additional assumption that object o_i is owned by agent i for all $i \in A$, then individual rationality is a natural additional constraint. In that case,

$$C(\succ) = \{m \in \mathcal{M} | o_i \succ_i o_j \Rightarrow m(i,j) = 0\}.$$

Yılmaz (2010) adapts the simultaneous-eating algorithm by imposing inequalities guaranteeing that it outputs an individually rational random matching. In addition to the sub-bistochasticity conditions (3), the following must also be true for any interim sub-random matrix

$$|U_T| - |T| \geq \sum_{i \in A \setminus T, j \in U_T} M^t(i,j) \text{ for all } T \subset A, \tag{4}$$

where $U_T \subset O$ is the set of objects that at least one agent in T finds acceptable (i.e. prefers over her initial endowment). Yılmaz's (2010) result implies that if a sub-random matrix satisfies these inequalities, then that matrix is in $\text{lcs}(\Delta C(\succ))$ because, as noted above, if this does not hold, the outcome of the algorithm would not satisfy individual rationality. The converse is also true. To see that, let $M \in \text{lcs}(\Delta C(\succ))$ and let M' be a bistochastic matrix representing an individually rational lottery with $M' \geq M$. For any $T \subset A$, the agents in T must have received only probability shares in U_T by individual rationality. The remainder of the probability shares of objects in U_T must be distributed among agents outside T. Since objects and agents have unitary supply and demand, respectively, the following must be true:

$$\sum_{i \in A \setminus T, j \in U_T} M'(i,j) = |U_T| - |T|.$$

Since $M(i,j) \leq M'(i,j)$, we have

$$\sum_{i \in A \setminus T, j \in U_T} M(i,j) \leq |U_T| - |T|.$$

Thus, in addition to the non-negativity constraints, (3) and (4) characterize the set $\text{lcs}(\Delta C(\succ))$.

Example 3. (Bihierarchical constraints.) Budish et al. (2013) introduce the class of *bihierarchical constraints* and generalize the probabilistic serial mechanism to those constraints. A set of bihierarchical constraints includes maximum quotas placed on all rows (agent demand) and all columns (object supply). In the case of single-unit demand and supply, these reduce to (3). Additionally, the constraints may also include constraints placed on some subcolumns such that for any j the constraints

$$\sum_{i \in A'} M(i,j) \leq b' \text{ and } \sum_{i \in A''} M(i,j) \leq b''$$

must satisfy either $A' \subset A''$, $A'' \subset A'$ or $A' \cap A'' = \varnothing$.

In the case of school choice, for example, the subcolumnar constraints can be interpreted as not allowing too many students with certain domicile neighborhoods or background characteristics into a given school. In the language of this paper, the constraint correspondence C is constant. For any $\succ$, the set $C(\succ)$ equals all matchings in $\mathcal{M}$ that are represented by permutation matrices that satisfy all the bihierarchical constraints. Analogously to Example 2, the results of Budish et al. (2013) imply that if an interim sub-random matrix satisfies the bihierarchical constraints, then it is in the set $\text{lcs}(\Delta C(\succ))$. The converse is straightforward as well: if $M \in \text{lcs}(\Delta C(\succ))$, then there exists some assignment matrix $M' \geq M$ that satisfies the bihierarchical constraints. Since $M \leq M'$ and all constraints are from the set Ω^0, M must satisfy them as well. This justifies the use of the bihierarchical constraints as algorithm inequalities.

Example 4. (Object allocation with minimum quotas.) Returning to BM's object-allocation setting with $n = k = 3$ with all objects having unitary capacity and all agents having unit demand, assume that there are two additional constraints on the possible ex-post deterministic matchings: object o_1 needs to be allocated to one of the agents 1 and 2, while object o_3 needs to be allocated to one of the agents 1 and 3. Formally, some $m \in \mathcal{M}$ is an element of $C(\succ)$ for all $\succ$ if and only if $m(1,1) + m(2,1) \geq 1$ and $m(1,3) + m(3,3) \geq 1$. Notice that this implies $m(3,1) = 0$ and $m(2,3) = 0$, which reduces dimensionality of the lower contour set of $C(\succ)$ and simplifies the problem of computing the inequalities defining it. Computing the additional inequalities implied by Proposition 1, we can find that the lower contour set of $C(\succ)$ is defined by the following two inequalities in addition to the usual bistochasticity conditions (3):[12]

$$M(1,1) + M(1,2) + M(3,2) \leq 1, \text{ and } M(1,2) + M(1,3) + M(2,2) \leq 1.$$

If agents have the following preferences

$$\succ_1: o_2 \succ_1 o_3 \succ_1 o_1, \qquad \succ_2: o_3 \succ_2 o_2 \succ_2 o_1, \qquad \succ_3: o_1 \succ_3 o_2 \succ_3 o_3,$$

then at the start of the GCPS algorithm, all three agents would claim shares from object o_2.

[12] See footnote 10 and Appendix B for more detail on the computational procedures used.

That object is the most preferred object for the first agent, while, as noted above, agents 2 and 3 are forbidden from claiming shares from their top choice. At time $t = 1/3$, the unit mass of probability shares of object o_2 has been completely claimed. Agents 1 and 3 switch to claiming shares from object o_3, while agent 2 claims shares from object o_1. At $t = 2/3$, we arrive at the following interim matrix:

$$M^{2/3} = \begin{pmatrix} 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 \end{pmatrix}.$$

At this point, the inequality $M(1,2) + M(1,3) + M(2,2) \leq 1$ starts binding: this fixes the value of $M(1,2)$ to $1/3$ for the rest of the algorithm, and makes o_2 unavailable to agent 1. This shifts 1 to claiming shares from object o_1 for the remainder of the algorithm. The final random allocation selected by the mechanism is

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & 1/3 & 0 \\ 0 & 1/3 & 2/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is clear that all three of the deterministic matchings in the support of the mechanism's outcome satisfy the outcome constraints (i.e. they are members of $C(\succ)$). In addition, it is easy to verify that the final allocation is constrained sd-efficient with respect to $\succ$.

Example 5. (Object allocation with relative-endowment constraints.) Consider another object-allocation example with $n = k = 3$, $d = 1$ and where $q_{o_1} = 2$ and $q_{o_2} = q_{o_3} = 1$. Assume that the only additional outcome constraint on the final allocation is that agent 1 receives at least as much from object o_1 as agent 2 does. (Equivalently, if 2 receives a copy of o_1 then 1 must also receive a copy of o_1.) Formally, $m \in \mathcal{M}$ is in $C(\succ)$ for all $\succ$ if and only if $m(1,1) \geq m(2,1)$. Computing the additional inequalities implied by Proposition 1, we can find that the lower contour set of $C(\succ)$ is defined by the following inequalities, in addition to the bistochasticity conditions (3):

$M(1,2) + M(1,3) + M(2,1) \leq 1;$

$M(1,2) + M(2,1) + M(2,2) \leq 1;$

$M(1,3) + M(2,1) + M(2,3) \leq 1;$

$M(2,1) + M(3,1) \leq 1;$ and

$M(1,2) + M(1,3) + M(3,2) + M(3,3) \leq 1.$

Assume that agents have the following preferences:

$$\succ_1: o_2 \succ_1 o_3 \succ_1 o_1, \qquad \succ_2: o_1 \succ_2 o_2 \succ_2 o_3, \qquad \succ_3: o_1 \succ_3 o_2 \succ_3 o_3.$$

Initially, agent 1 claims shares from o_2, while agents 2 and 3 claim shares from o_1. The interim matrix corresponding to time $t = 1/2$ is:

$$M^{1/2} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}.$$

At this stage, the inequalities

$$M(1,2)+M(1,3)+M(2,1) \leq 1;\ M(1,2)+M(2,1)+M(2,2) \leq 1 \text{ and } M(2,1)+M(3,1) \leq 1$$

all bind. In particular, this makes o_1 unavailable to agents 2 and 3, o_2 unavailable to 1 and 2, and o_3 unavailable to 1. For the rest of the algorithm, agents 1 and 2 have a single available object remaining: o_1 for 1 and o_3 for 2. Both o_2 and o_3 are available to agent 3 so she claims shares from o_2, her more preferred of the two. The final random allocation and its decomposition are:

$$\tilde{M} := M^1 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

While the constrained sd-efficiency of $\tilde{M}$ follows from Proposition 2, directly verifying that fact is not immediate. For the full argument, see the proof of the following in the Appendix.

Proposition 3. *The random allocation represented by $\tilde{M}$ is constrained sd-efficient among all lotteries putting positive probability only on allocations satisfying $m(1,1) \geq m(2,1)$.*

Example 6. (Roommate problem with individual rationality.) If $A = O$, $d = q_o = 1$ for all $o \in O$, and C equals

$$C(\succ) = \{m \in \mathcal{M} | \forall i,j \in A : m(i,j) = m(j,i); i \succ_i j \text{ or } j \succ_j i : m(i,j) = 0\},$$

then the model is isomorphic to the roommate problem (Gale and Shapley 1962) with individual rationality as a desideratum. Preferences over the set O are the preferences over potential roommates.[13]

The set of constraints characterizing the polytope $\text{lcs}(\Delta C(\succ))$ for $n = 3$ are $M(i,j) \leq 0$ if $i \succ_i j$ or $j \succ_j i$, the sub-bistochasticity conditions (3), as well as

$$\begin{aligned} &M(1,2)+M(1,3)+M(2,3)\leq 1; \qquad && M(1,2)+M(1,3)+M(3,2)\leq 1;\\ &M(1,2)+M(3,1)+M(2,3)\leq 1; && M(1,2)+M(3,1)+M(3,2)\leq 1;\\ &M(2,1)+M(1,3)+M(2,3)\leq 1; && M(2,1)+M(1,3)+M(3,2)\leq 1;\\ &M(2,1)+M(3,1)+M(2,3)\leq 1; && M(2,1)+M(3,1)+M(3,2)\leq 1. \end{aligned}$$

Assume that the preference profile $\succ$ is

$$\succ_1: 2 \succ_1 3 \succ_1 1; \qquad \succ_2: 1 \succ_2 3 \succ_2 2; \qquad \succ_3: 1 \succ_3 2 \succ_3 3.$$

All agents are available to each other at the beginning of the algorithm. Agent 1 claims probability shares from 2, while 2 and 3 claim shares from 1. At $t = 1/2$, the interim sub-random matrix is

$$M^{1/2} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}.$$

The following inequalities all bind at $t = 1/2$:

$$\begin{aligned} &M(1,2)+M(3,1)+M(2,3)\leq 1; \qquad && M(1,2)+M(3,1)+M(3,2)\leq 1;\\ &M(2,1)+M(3,1)+M(2,3)\leq 1; && M(2,1)+M(3,1)+M(3,2)\leq 1. \end{aligned}$$

Thus, according to the GCPS procedure, the coordinates $(1,2)$, $(2,1)$, $(2,3)$, $(3,1)$, and $(3,2)$ are all fixed at their current values and all three agents continue claiming probability shares from their next most preferred roommate. For 1, this is 3 (as $(1,3)$ has not been fixed yet), while both 2 and 3 can only claim probability shares from themselves, increasing the probability that they are left unmatched. The reason is that, in addition to $(1,2)$ and $(1,3)$, the coordinates $(2,3)$ and $(3,2)$ were also fixed, making 2 and 3 unavailable to each other.

[13] Alternatively, the same model represents the problem of pairwise kidney exchange: instead of individual agents, each element of A is a patient-donor pair (Roth et al. 2005a). Preferences in this case model the relative compatibility (or the projected five-year survival rate) of different kidneys coming from the living donors in each patient-donor pair: e.g. $i \succ_\ell j$ implies that donor i's kidney has a better expected long-term outcome for patient ℓ than donor j's kidney, while $\ell \succ_\ell i$ implies that donor i's kidney is not transplantation-compatible for patient ℓ (Nicoló and Rodríguez-Álvarez 2012, 2013, 2017, Balbuzanov 2018).

The final random matching selected by the mechanism is

$$GCPS(\succ, \Omega^{C(\succ)}) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

From the decomposition, it is clear that the lottery assigns an equal probability to 1 and 2 being matched as roommates (while 3 is left unmatched), and to 1 and 3 being matched as roommates (while 2 is left unmatched). It is easy to verify that $GCPS(\succ, \Omega^{C(\succ)})$ is constrained sd-efficient.

Example 7. (Two-sided matching with stability as a constraint.)

If, in addition to the assumptions from Example 6, A can be partioned into the disjoint sets $\{A_1, A_2\}$ such that

$$\begin{aligned} C(\succ) = \{m \in \mathcal{M} | \forall i, j \in A : m(i,j) = m(j,i); \\ i \succ_i j \text{ or } j \succ_j i \text{ or } i, j \in A_k, k \in \{1,2\}, i \neq j : m(i,j) = 0\}, \end{aligned}$$

the model now corresponds to one-to-one two-sided matching (Gale and Shapley 1962) where A_1 and A_2 represent the two sides of the market with agents on each side able to be matched only with agents on the other side.

Consider the following two-sided one-to-one matching problem with the additional requirement that any random matching places positive probability only on stable allocations. Let the three men (m_1, m_2, m_3) and three women (w_4, w_5, w_6) have the following preferences[14]:

$$\begin{aligned} &\succ_{m_1} : w_4 \succ_{m_1} w_6 \succ_{m_1} w_5 \succ_{m_1} m_1; \qquad &&\succ_{w_4} : m_2 \succ_{w_4} m_1 \succ_{w_4} m_3 \succ_{w_4} w_4; \\ &\succ_{m_2} : w_5 \succ_{m_2} w_6 \succ_{m_2} w_4 \succ_{m_2} m_2; &&\succ_{w_5} : m_3 \succ_{w_5} m_2 \succ_{w_5} m_1 \succ_{w_5} w_5; \\ &\succ_{m_3} : w_6 \succ_{m_3} w_5 \succ_{m_3} w_4 \succ_{m_3} m_3; &&\succ_{w_6} : m_1 \succ_{w_6} m_2 \succ_{w_6} m_3 \succ_{w_6} w_6. \end{aligned}$$

The three stable allocations with respect to these preferences are:

$$\mu = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_4 & w_5 & w_6 \end{pmatrix}; \ \sigma = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_4 & w_6 & w_5 \end{pmatrix}, \text{ and } \nu = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_6 & w_4 & w_5 \end{pmatrix}.$$

We can represent each allocation as a bistochastic 6×6 matrix in which the (i, j)-th coordinate is 1 if and only if the agent with subscript i and the agent with subscript j are matched

[14]This example is from Knuth (1997, p. 59).

with each other. The full list of the necessary algorithm inequalities is in Appendix B. At the start of the GCPS algorithm, each agent claims probability shares from his or her most preferred partner. At $t = 1/2$, the interim matrix is

$$M^{1/2} = \begin{pmatrix} 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

At this time, a number of the inequalities from $\Omega^{\Delta C(\succ)}$ begin binding and the values of the entries with coordinates $(1,4)$, $(2,5)$, $(2,6)$, $(3,6)$, $(4,2)$, $(5,3)$, $(6,1)$, and $(6,2)$ are fixed for the rest of the algorithm. This means that m_1 can only claim probability shares from w_6: w_6 is the unique agent available to m_1 after time $t = 1/2$. Similarly, each of the other agents has a unique other agent from whom he or she can claim probability shares: they are w_4, w_5, m_1, m_2, and m_3 for m_2, m_3, w_4, w_5, and w_6, respectively. The final random allocation and its decomposition are:

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\mu + \frac{1}{2}\nu.$$

As a convex combination of the men-optimal and women-optimal stable matchings, it is easy to verify that $\frac{1}{2}\mu + \frac{1}{2}\nu$ is also constrained sd-efficient with respect to the preference profile $\succ$.

5 Conclusion

This paper generalizes the Probabilistic Serial mechanism to an arbitrary matching setting with arbitrary constraints. Regardless of the set of allowable ex-post allocations, the GCPS mechanism selects an sd-efficient lottery over that set and, furthermore, does so in an anonymous way if the constraints allow it. The settings include one-sided and two-sided matching markets. The constraints can include individual rationality, exchange-cycle constraints, maximal-cardinality matchings, various quotas and/or caps etc. They can also be merely an arbitrary list of permissible matchings.

Understanding the properties of the general mechanism presented here is a valuable further avenue of research. Examples include finding a natural fairness property it satisfies, especially when paired with a constraint correspondence that is not anonymous. Also, the GCPS embeds the PS and the GPS (Budish et al. 2013) mechanisms as special cases, both of which are weakly and convexly strategyproof (Balbuzanov 2016). However, it also includes PS variants, where weak strategyproofness fails (Yılmaz 2010). A natural question to ask is: what conditions guarantee weak or convex strategyproofness? Finally, while some outcome constraints can serve as algorithm inequalities without jeopardizing sd-efficiency (e.g. Examples 1 and 3), in most of the cases considered above the outcome constraints (even if expressible as positive linear inequalities) are not appropriate algorithm inequalities. Characterizing the outcome constraints that can be used as algorithm inequalities could provide a useful insight into the PS algorithm.

A Proofs

Proof of Proposition 1: Fix a preference profile $\succ$. Let the set of permissible ex-post deterministic matchings be $C := \{M_1, \ldots, M_p\}$, where the argument of $C(\succ)$ is omitted for notational ease. Denote the lower contour set of $\text{co}(C)$ by D. It is clear that $\text{co}(C) \subset D$.

Claim 1. *The convex hull of the set $\mathcal{E}_i$, which is defined by*

$$\mathcal{E}_i := \{M \in \mathbb{R}_+^{n\times k} | \forall (a,o) \in A \times O : M(a,o) = 0 \textit{ or } M(a,o) = M_i(a,o)\},$$

equals the set $\{M \in \mathbb{R}_+^{n\times k} | M \leq M_i\}$.

Proof of Claim 1: If $M \in \mathcal{E}_i$, then $M \leq M_i$. This relationship is preserved under convex combinations and so $\text{co}(\mathcal{E}_i) \subseteq \{M \in \mathbb{R}_+^{n\times k} | M \leq M_i\}$. Showing the converse set inclusion is done via induction. The base step is to see that for any matrix coordinate (a, o) the following set is contained in $\text{co}(\mathcal{E}_i)$:

$$\{M \in \mathbb{R}_+^{n\times k} | \forall (a',o') \in A \times O \backslash \{(a,o)\} : M(a',o') = 0 \text{ or } M(a',o') = M_i(a',o'); M(a,o) \in [0, M_i(a,o)]\}.$$

The inductive step is to observe that if the following set is contained in $\text{co}(\mathcal{E}_i)$ for any proper subset $S \subsetneq A \times O$

$$\{M \in \mathbb{R}_+^{n\times k} | \forall (a,o) \notin S : M(a',o') = 0 \text{ or } M(a',o') = M_i(a',o'); \forall (a,o) \in S : M(a,o) \in [0, M_i(a,o)]\},$$

then the same must hold true for any $S' \subseteq A\times O$ with $S' = S\cup\{a,o\}$ with $(a,o) \in (A\times O)\backslash S$. Ultimately:

$$\mathrm{co}(\mathcal{E}_i) \supseteq \{M \in \mathbb{R}_+^{n\times k} | \forall (a,o) \in A\times O : M(a,o) \in [0, M_i(a,o)]\} = \{M \in \mathbb{R}_+^{n\times k} | M \leq M_i\},$$

which is what we wanted to show. □

Claim 2.

$$D = \mathrm{co}\left(\bigcup_{i=1}^{p} \mathcal{E}_i\right).$$

Proof of Claim 2: Let $M \in \mathrm{co}\left(\cup_{i=1}^{p}\mathcal{E}_i\right)$. Equivalently:

$$M = \sum_{i=1}^{q} \pi_i M^i,$$

where $\sum_{i=1}^{q} \pi_i = 1$ and $\pi_i \geq 0$ for all i. Also for each M^i, there exists $j \in \{1,\ldots,p\}$ such that $M^i \in \mathcal{E}_j$ and so there exists a corresponding matrix $\overline{M}^i \in C$ such that $\overline{M}^i \geq M^i$. Then

$$M \leq \sum_{i=1}^{q} \pi_i \overline{M}^i \in \mathrm{co}(C).$$

As D is the lower contour set of $\mathrm{co}(C)$, it follows that $M \in D$.

Now assume that $M \in D$. Then there exists some $M' \in \mathrm{co}(C)$ with $M' = \sum_{i=1}^{p} \pi_i M_i$ such that $M' \geq M$. For each $\alpha \in \{1,\ldots,n\}, \beta \in \{1,\ldots,k\}$ define:

$$\gamma(\alpha,\beta) = \begin{cases} 0 & \text{if } M(\alpha,\beta) = 0, \\ M(\alpha,\beta)/M'(\alpha,\beta) & \text{otherwise.} \end{cases}$$

Define $\underline{M}_i$ entry by entry via $\underline{M}_i(\alpha,\beta) = \gamma(\alpha,\beta)M_i(\alpha,\beta)$. Clearly, $\underline{M}_i \leq M_i$ and so by Claim 1

$$\underline{M}_i \in \mathrm{co}(\mathcal{E}_i) \subset \mathrm{co}\left(\bigcup_{i=1}^{p} \mathcal{E}_i\right).$$

It is straightforward to verify that $M = \sum_{i=1}^{p} \pi_i \underline{M}_i$. Thus, $M \in \mathrm{co}\left(\cup_{i=1}^{p}\mathcal{E}_i\right)$. □

Claim 2 states that D is the convex hull of finitely many points and, hence, a bounded convex polytope in $\mathbb{R}^{n\times k}$. So D can be represented as the intersection of finitely many closed halfspaces in $\mathbb{R}^{n\times k}$.

Some matrix entries might be zero for all of the elements of D. Let there be q such entries. Their corresponding positive linear inequalities take the form $M(\cdot,\cdot) \leq 0$. For the remainder of the proof, consider D as a subset of $\mathbb{R}^{nk-q}$ rather than $\mathbb{R}^{n\times k}$ by disregarding all coordinates (a, o), for which $M(a, o) = 0$ for all $M \in D$.

Claim 3. *The polytope D is fully dimensional when viewed in $\mathbb{R}^{nk-q}$ or, equivalently, D has a non-empty (topological) interior.*[15]

Proof of Claim 3: Consider the following element of D:

$$N = \sum_{i=1}^{p} \frac{1}{p} M_i.$$

Note that all of N's coordinates are positive because for all $nk - q$ coordinates, there exists some M_i for which that coordinate is greater than zero. Also it is clear that $N \in \text{co}(C) \subset D$ and, because $\frac{1}{2}N < N$, it follows that $\frac{1}{2}N \in D$. Then, for every sufficiently small $\varepsilon > 0$, every N' in the open ball with radius ε around $\frac{1}{2}N$ satisfies the following two conditions: $N' \geq 0$ and $N' \leq N$. This implies $N' \in D$. Thus $\frac{1}{2}N$ lies in the interior of D and D is fully dimensional in $\mathbb{R}^{nk-q}$. □

Since the convex polytope D is fully dimensional in $\mathbb{R}^{nk-q}$, it can be uniquely minimally defined (up to rescaling by a positive scalar) as all elements x in $\mathbb{R}^{nk-q}$ that satisfy $Ax \leq b$ for some matrix A and a vector b. Write each of the individual inequalities as $A_i \cdot x \leq b_i$, where A_i is a row in A. Note that since D contains $\mathbf{0}$, $b_i \geq 0$ holds for all i. As D is entirely contained in the non-negative orthant of $\mathbb{R}^{nk-q}$, the set of inequalities must include non-negativity constraints for the coordinates of all $x \in D$: the constraints would correspond to the hyperplanes bounding the non-negative orthant and would satisfy $A_i \leq 0$ and $b_i = 0$. Conversely, if some A_i satisfies $A_i \leq 0$, then the corresponding b_i must equal zero. Otherwise, that constraint would be strictly weaker than the non-negativity constraint and would thus violate the minimality of the constraint set.

Claim 4. *Any inequality $A_i \cdot x \leq b_i$ that does not correspond to a non-negativity constraint satisfies $A_i \geq 0$.*

Proof of Claim 4: Towards contradiction assume that some coordinates of A_i are strictly positive and some strictly negative. The polytope D is fully dimensional (Claim 3). By the full dimensionality of D, there exists some $y \in D$ such that $A_i \cdot y = b_i$ but $A_j \cdot y < b_j$ for all $j \neq i$.[16] Then one can find some $x \neq z$ distinct from y such that $z \geq x \geq y$ with $A_i \cdot z = b_i$

[15]See, for example, Section 2.3 in Ziegler (2007) for the relevant definitions and equivalency result.

[16]See, for example, Corollary 8.2a in Schrijver (1986). It states that given a fully dimensional polytope D and a constraint $A_i \cdot x \leq b_i$ defining it, then for each other constraint $A_j \cdot x \leq b_j$, there exists some $x_j \in D$ such that $A_i \cdot x_j = b_i$ but $A_j \cdot x_j < b_j$. Then an equal-weight convex combination of all x_j's would satisfy the desired conditions.

and $A_j \cdot z < b_j$ for all $j \neq i$, while $A_i \cdot x > b_i$.[17] In other words, $z \in D$ while $x \notin D$. But by the way D was defined, there exists some $w \in \text{co}(C)$ such that $w \geq z$ and, since $z \geq x$, $w \geq x$ must hold and hence $x \in D$. Contradiction! Therefore all constraints, other than the ones guaranteeing non-negativity, are positive linear inequalities. That is, they take the form

$$\sum_{(i,j)\in A\times O} a(i,j)M(i,j) \leq b$$

for some $b > 0$ with $a(i,j) \geq 0$ for all (i,j). □

Claim 4 completes the proof of Proposition 1.

□

Proof of Proposition 2: The notation from the proof of Proposition 1 is carried through to this proof: in particular, the set C is fixed and $D := \text{lcs}(\text{co}(C))$. Recall from Definition 1 that M^t denotes the sub-random matrix arising at time t of the GCSE algorithm. Additionally, an object o_j is said to be available to agent i at time t if none of the constraints from Ω^C (for which $a(i,j) > 0$) binds at time t.

Claim 5. *For any time $t < 1$ of the GCSE algorithm and all $i \in A$, either the set of objects available to i at t is non-empty or i has fulfilled her demand at t (i.e. $\sum_{j=1}^{j=k} M^t(i,j) = d$). The GCSE algorithm terminates at $t = 1$ and its output (i.e. the matrix M^1) is in* $\text{co}(C)$.

Proof of Claim 5: By construction of the algorithm, M^t satisfies all inequalities in Ω^C for all $t \leq 1$. As Ω^C characterizes the polytope D, this implies $M^t \in D$ and, therefore, there exists $M' \in \text{co}(C)$ such that $M^t \leq M'$.

Any $m \in C$ satisfies $\sum_{j=1}^{j=k} m(i,j) = d$ for all $i \in A$ and so M' satisfies it as well. As $\int_0^1 e_i(t)dt = d$, it follows that $\sum_{j=1}^{j=k} M^t(i,j) = \int_0^t e_i(t)dt \leq d$ for all i. Therefore, $\sum_{j=1}^{j=k} M^t(i,j) \leq \sum_{j=1}^{j=k} M'(i,j)$ for all i. This, together with $M^t \leq M'$, implies that for every i either there exists $o_j \in O$ such that $M^t(i,j) < M'(i,j)$ or $\sum_{j=1}^{j=k} M^t(i,j) = d$. If $M^t(i,j) < M'(i,j)$ for some o_j and i, there exists $\varepsilon > 0$ such that $M^t(i,j)+\varepsilon J(i,j) < M'(i,j)$ where $J(i',j') = \mathbb{1}_{\{(i,j)\}}(i',j')$ and $\mathbb{1}_{\{(i,j)\}}$ is the indicator function for the set $\{(i,j)\}$. Thus, $M^t(i,j)+\varepsilon J(i,j) \in D$ and so i claiming o_j cannot violate any inequalities from Ω^C. In other words, o_j is available to i at time t. This settles the first half of the claim.

[17] Assuming without loss of generality that the first coordinate of A_i (call it A_i') is strictly positive, while the second one (A_i'') is strictly negative, x can be derived from y by increasing its first coordinate by some small $\varepsilon > 0$, and z from x by increasing its second coordinate by $\frac{A_i'}{|A_i''|}\varepsilon$. Then clearly $z \geq x \geq y$ and $A_i \cdot x > A_i \cdot z = b_i$. For all small enough ε, z is close enough to y so that all inequalities corresponding to $j \neq i$ would be preserved strictly.

In particular, this implies that M^t for all $t \leq 1$ exists and are well defined. Consider M^1: it satisfies all inequalities in Ω^C implying $M^1 \in D$. Thus there exists $M' \in \text{co}(C)$ such that $M^1 \leq M'$. Summing across all agents, we have $\sum_{i \in A, j \in \{1,2,\ldots,k\}} M'(i,j) = nd$. The same equation holds for M^1 as well because $\int_0^1 e_i(t)dt = d$. Thus $M^1 = M' \in \text{co}(C)$, which is what we wanted to show.

□

It is clear that, by giving all agents the same claiming-speed functions, the GCPS is anonymous if the constraint correspondence C is itself anonymous. It remains to be shown that the GCSE algorithm is C-constrained sd-efficient:

Claim 6. *If M^1 is weakly Pareto dominated with respect to first-order stochastic dominance by some $M' \in \text{co}(C)$, then $M^1 = M'$.*

Proof of Claim 6: Let T be the set of times at which an inequality $(a,b) \in \Omega^C$ starts binding and such that there exists some (i,j) with $a(i,j) > 0$ and $M^1(i,j) \neq M'(i,j)$. The set T is clearly finite and we can set $t^* = \min T$.

Thus, at time t^*, the inequality (a,b) starts binding and object o_j becomes unavailable to agent i. Note that M^t being non-decreasing in t by construction and the fact that $a \geq 0$ together guarantee that if (a,b) binds at time t^*, it binds for all $t \in [t^*, 1]$. Therefore:

$$\sum_{(i',j') \in A \times O} a(i',j') M^1(i',j') = b.$$

Since $M' \in \text{co}(C) \subset D$, we also have

$$\sum_{(i',j') \in A \times O} a(i',j') M'(i',j') \leq b.$$

So

$$\sum_{(i',j') \in A \times O} a(i',j') M^1(i',j') \geq \sum_{(i',j') \in A \times O} a(i',j') M'(i',j').$$

We can assume without loss of generality that $M^1(i,j) > M'(i,j)$ since if $M^1(i,j) < M'(i,j)$, the preceding inequality implies that there exists some other pair $(i'',j'') \in A \times O$ which satisfies $M^1(i'',j'') > M'(i'',j'')$ and $a(i'',j'') > 0$ and, therefore, $o_{j''}$ would become unavailable to agent i'' also at time t^*.

Since agent i strictly prefers $M'(i)$ over $M^1(i)$ in first-order stochastic dominance sense, there must exist an object o_l such that, while $M^1(i,j) > M'(i,j) \geq 0$,

$$0 \leq M^1(i,l) < M'(i,l) \text{ and } l \succ_i j.$$

Then, since $M^1(i,j) > 0$, agent i must have claim probability shares from o_j during the mechanism even though she prefers object o_l. So when i was consuming j's probability shares (which happens before t^*), o_l must have already become unavailable to i. So l must have become unavailable to agent i strictly before t^*. But $M^1(i,l) \neq M'(i,l)$, which contradicts the choice of t^*. □

□

Proof of Proposition 3. : Toward contradiction, assume that there exists $M \in \Delta C(\succ)$ that weakly FOSD-dominates $\tilde{M}$ for all agents and strictly for some. Without loss of generality, assume M is constrained sd-efficient.

First note that any deterministic allocation m for which $m(1,1) = 1$ and $m(2,1) = 0$ is Pareto inefficient. If $m(3,1) = 1$, agent 1's assignment can be improved without changing the allocation of the two other agents by giving her the object in $\{o_2, o_3\}$ that is not assigned to agent 2. If $m(3,1) = 0$ instead, the second copy of o_1 can be assigned to agent 3, improving her outcome without changing the allocation of the other two agents. As M is sd-efficient, it can place positive probability only on allocations that are Pareto efficient. Thus, M cannot place positive probability on any deterministic m, for which $m(1,1) = 1$ and $m(2,1) = 0$. As the outcome constraint in this problem is $m(1,1) \geq m(2,1)$, this implies that M can place positive probability only on allocations m for which $m(1,1) = m(2,1) = 0$ or $m(1,1) = m(2,1) = 1$ and so $M(1,1) = M(2,1)$.

This, in turn, implies that $M(1,1) + M(3,1) = M(2,1) + M(3,1) \leq 1$ as o_1, having capacity $q_{o_1} = 2$, can be assigned either to *both* of 1 and 2 or *only* to agent 3. As o_1 is agent 3's most preferred object, efficiency requires $M(1,1) + M(3,1) = M(2,1) + M(3,1) = 1$.

The lottery M FOSD-dominates $\tilde{M}$ so at least one of the three agents i strictly prefers $M(i)$ in FOSD sense to $\tilde{M}(i)$. Agent 1 can be made better off in FOSD sense when moving from $\tilde{M}$ to M only if $M(1,1) < \tilde{M}(1,1) = 1/2$. The fact that $M(1,1) = M(2,1)$ implies that $M(2,1) < 1/2 = \tilde{M}(2,1)$ and so agent 2 cannot be weakly better off under M relative to $\tilde{M}$ if agent 1 is strictly better off.

If agent 3 strictly prefers $M(3)$ to $\tilde{M}(3)$, this implies that $M(3,1) > \tilde{M}(3,1) = 1/2$ as $\tilde{M}$ gives agent 3 a convex combination of her most-preferred and second most-preferred objects. We established that $M(2,1) + M(3,1) = 1$ and so $M(3,1) > \tilde{M}(3,1) = 1/2$ implies that $M(2,1) < \tilde{M}(2,1) = 1/2$ and so agent 2 cannot be weakly better off under M relative to $\tilde{M}$ if agent 3 is strictly better off.

Finally, if agent 2 strictly prefers $M(2)$ to $\tilde{M}(2)$, this implies either that $M(2,1) > \tilde{M}(2,1) = 1/2$ or $M(2,2) > \tilde{M}(2,2) = 0$. If $M(2,1) > \tilde{M}(2,1) = 1/2$, the equality

$M(2,1) + M(3,1) = 1$ implies that $M(3,1) < \tilde{M}(3,1) = 1/2$ and so agent 3 cannot be weakly better off under M relative to $\tilde{M}$ if agent 2 is strictly better off.

If $M(2,2) > \tilde{M}(2,2) = 0$ instead, then either $M(1,2) < \tilde{M}(1,2) = 1/2$ (making it impossible for 1 to be weakly better off under M relative to $\tilde{M}$) or $M(3,2) < \tilde{M}(3,2) = 1/2$. As 3 weakly prefers M to $\tilde{M}$, this necessitates $M(3,1) > \tilde{M}(3,1)$ which, as shown above, also leads to a contradiction.

Thus, it is impossible for all agents to weakly (and strictly for some) prefer M in FOSD-sense to $\tilde{M}$. Thus $\tilde{M}$ is sd-efficient. □

B Computing Ω^C

One of the goals of this paper is to guide users who want to use a version of the PS mechanism for their constrained environment on how to compute the inequalities defining the lower contour set of ΔC for a given list of allowable allocations C. As shown in the proof of Proposition 1, the lower contour set of ΔC equals the convex hull of the points in the following set:

$$\mathcal{E} := \bigcup_{M_i \in C} \{M \in \mathbb{R}_+^{n \times k} | \forall (a,o) \in A \times O : M(a,o) = 0 \text{ or } M(a,o) = M_i(a,o)\}.$$

In many cases, $\mathcal{E}$ is also the set of extreme points (or vertices) of lcs(ΔC).[18] To find the elements of $\mathcal{E}$, one can iteratively take matrices M representing elements in C and find all matrices each of whose entries equals either zero or the corresponding entry in M. Thus, if M has k non-zero entries, M can contribute up to 2^k elements to $\mathcal{E}$, including the zero matrix and M itself. Of course, the marginal contribution to the cardinality of $\mathcal{E}$ for each subsequent element of C will be much smaller than that as many of the extreme points corresponding to it would also correspond to other elements of C.

As an illustration, let us revisit Example 7. In that example, the outcome constraint is stability: all random matchings must place positive probability only on stable allocations. Given the preferences in the problem, there are three stable allocations and therefore three

[18] I.e. for each $E \in \mathcal{E}$, there are no $E', E'' \in$ lcs(ΔC) (both distinct from E) such that E is a convex combination of E' and E''. If no agent is permitted to consume multiple different quantities of a single object (which is true if, for example, all agents have unit demand), $\mathcal{E}$ is the set of extreme points of lcs(ΔC). If, instead, agent a can feasibly consume, say, one or two copies from object o, then $\mathcal{E}$ contains the matrices defined by $M'(i,j) = \mathbb{1}_{\{(a,o)\}}(i,j)$ and $M''(i,j) = 2 \times \mathbb{1}_{\{(a,o)\}}(i,j)$. Then M' is not an extreme point as it is a convex combination of M'' and the zero matrix. Removing all interior points (i.e. all matrices similar to M') from $\mathcal{E}$ to arrive at the set of extreme points is straightforward, however.

elements of $C(\succ)$:

$$\mu = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}; \sigma = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \text{ and } \nu = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It can be checked that the corresponding set of extreme points $\mathcal{E}$ contains 184 elements, including μ, σ, and ν, and the zero matrix.[19]

Once the set of extreme points defining $\text{lcs}(\Delta C)$ is known, the next step is computing the inequalities corresponding to the hyperplanes defining $\text{lcs}(\Delta C)$. This is known as the *facet-enumeration problem* (FEP). One can use specialized computational packages such as `cdd` (Fukuda 1999), `lrs` (Avis 2000), or Qhull (Barber et al. 1996) for this. These packages are implementations of different theoretical algorithms: `cdd` is based on Motzkin et al.'s (1953) double description method, `lrs` implements Avis and Fukuda's (1992) reverse search, and Qhull uses ideas from Clarkson and Shor's (1989) randomized and Kallay's beneath-beyond algorithms (Preparata and Shamos 1985).[20] See Joswig and Lorenz (2018) for a survey of other available specialized software.

While algorithms for the FEP have worst-case complexity that is non-polynomial in the number of extreme points and the number of dimensions (Seidel 2018)[21], practical experience suggests that computation is generally fast. Computing the 62 facets of the polytope arising in the stability example (184 extreme points in 14 dimensions) using version 0.61 of `cdd` is instantaneous on a superannuated ultrabook. As an additional computational experiment, consider a more complex version of Example 6 by moving to four agents and allowing exchange cycles of length no more than 3.[22] Computing the 538 facets of the 16-dimensional polytope with 179 vertices that arises here takes `cdd` one second of processor time on the same computer.

For more computational experiments, see Avis et al. (1997) and Avis and Jordan (2018). Joswig and Lorenz (2018) summarize the findings of these and related papers by noting that

[19] The full list of extreme points is available from the author upon request.

[20] Both `lrs` and `cdd` have MATLAB interfaces. Curiously, `lrs` has found another economic application in the enumeration of all Nash equilibria of two-player normal-form games (Avis et al. 2010).

[21] In certain special cases, such as unit demand, when all lower contour sets of interest are 0/1-polytopes, there exists the possibility of defining specialized algorithms with good performance (potentially even running in polynomial time). Similarly, for specific classes of constraint correspondences, there may exist general closed-form sets of algorithm inequalities as in Yılmaz (2010) and Balbuzanov (2018).

[22] The details are available from the author upon request.

"[e]ssentially for each known algorithm there is a family of polytopes for which the given algorithm is superior to any other, and there is a second family for which the same algorithm is inferior to any other." Thus, software packages differ significantly in their performance and suitability for different classes of polytopes as they are implementations of different theoretical algorithms. For example, algorithms based on Motzkin et al.'s (1953) double description method (such as `cdd`) perform well with non-simplicial polytopes: i.e. polytopes whose faces are not all simplices. Even though the double description method is slower in terms of worst-case running time (Seidel 2018), it is appropriate for the computations considered here as it is easy to show that the lower contour sets are generically non-simplicial.

As a matter of practical consideration, one should reduce the dimensionality of the C-matrices by eliminating all coordinates which never differ from 0 before inputting the extreme points into the computational package in order to avoid errors and speed up computations. For example, in the stability example above, all coordinates

$$(i,j) \in (\{1,2,3\} \times \{1,2,3\}) \cup (\{4,5,6\} \times \{4,5,6\}) \cup \{(1,5),(3,4),(4,3),(5,1)\}$$

are zero for all elements in $\{\mu, \sigma, \nu\}$. The GCPS algorithm would fix these coordinates to be zero throughout its run. Thus, we can reduce the dimensionality of the problem by removing them and bringing the representation of the three stable allocations $\{\mu, \sigma, \nu\}$ and the extreme points they generate down to 14-dimensional vectors.

The output of these computational packages includes all non-negativity, row-capacity, and column-capacity constraints as well as all positive linear inequalities comprising Ω^C. Thus, the 184 extreme points of the stability example generate the 14 non-negativity inequalities for each of the 14 coordinates in the reduced-dimensionality problem: $M(i,j) \geq 0$ for all $(i,j) \in (\{1,2,3\} \times \{4,5,6\}) \cup (\{4,5,6\} \times \{1,2,3\}) \setminus \{(1,5),(3,4),(4,3),(5,1)\}$.

The inequalities generated also include all row and column constraints. For example,

$$M(1,4) + M(1,6) \leq 1$$

corresponds to the demand constraint of m_1: agent m_1 can be matched to exactly one other agent. The sum of these two coordinates suffices as the other coordinates in that row are fixed to zero. The remaining, substantive inequalities are as follows:

$$M(1,4) + M(4,2) \leq 1; \qquad M(2,4) + M(4,1) \leq 1;$$
$$M(1,4) + M(6,1) \leq 1; \qquad M(4,1) + M(1,6) \leq 1;$$
$$M(2,5) + M(5,3) \leq 1; \qquad M(5,2) + M(3,5) \leq 1;$$

$$M(3,6) + M(5,3) \leq 1; \qquad M(6,3) + M(3,5) \leq 1;$$
$$M(1,6) + M(2,5) + M(2,6) \leq 1; \qquad M(1,6) + M(2,5) + M(6,2) \leq 1;$$
$$M(1,6) + M(5,2) + M(2,6) \leq 1; \qquad M(1,6) + M(5,2) + M(6,2) \leq 1;$$
$$M(6,1) + M(2,5) + M(2,6) \leq 1; \qquad M(6,1) + M(2,5) + M(6,2) \leq 1;$$
$$M(6,1) + M(5,2) + M(2,6) \leq 1; \qquad M(6,1) + M(5,2) + M(6,2) \leq 1;$$
$$M(2,4) + M(2,6) + M(3,6) \leq 1; \qquad M(2,4) + M(2,6) + M(6,3) \leq 1;$$
$$M(2,4) + M(6,2) + M(3,6) \leq 1; \qquad M(2,4) + M(6,2) + M(6,3) \leq 1;$$
$$M(4,2) + M(2,6) + M(3,6) \leq 1; \qquad M(4,2) + M(2,6) + M(6,3) \leq 1;$$
$$M(4,2) + M(6,2) + M(3,6) \leq 1; \qquad M(4,2) + M(6,2) + M(6,3) \leq 1;$$
$$M(1,6) + M(2,6) + M(6,3) \leq 1; \qquad M(1,6) + M(6,2) + M(6,3) \leq 1;$$
$$M(1,6) + M(6,2) + M(3,6) \leq 1; \qquad M(6,1) + M(2,6) + M(3,6) \leq 1;$$
$$M(6,1) + M(2,6) + M(6,3) \leq 1; \qquad M(6,1) + M(6,2) + M(3,6) \leq 1;$$
$$M(2,4) + M(2,5) + M(6,2) \leq 1; \qquad M(2,4) + M(5,2) + M(6,2) \leq 1;$$
$$M(2,4) + M(5,2) + M(2,6) \leq 1; \qquad M(4,2) + M(2,5) + M(6,2) \leq 1;$$
$$M(4,2) + M(2,5) + M(2,6) \leq 1; \qquad M(4,2) + M(5,2) + M(2,6) \leq 1.$$

A natural next question to ask is: what is the total number of inequalities that can define our polytopes of interest? As each inequality defines a facet of the polytope, McMullen's Upper Bound Theorem (McMullen 1970) gives an answer to that question.

Theorem 1 (McMullen's Upper Bound Theorem)**.** *The total number of facets of a d-dimensional polytope with x vertices is no more than the following sum of binomial coefficients:*

$$\binom{x - \lceil d/2 \rceil}{\lfloor d/2 \rfloor} + \binom{x - \lceil (d+1)/2 \rceil}{\lfloor (d-1)/2 \rfloor}.$$

While the upper bound from McMullen's theorem is indeed tight (i.e. there exist d-polytopes with x vertices with a number of facets equal to the upper bound), in practice it is far larger than the actual number of facets in the problems of interest here. The 14-dimensional polytope in the stability example considered above has 184 extreme points implying a maximum number of $995,538,160,320$ facets. The polytope actually has 62 facets. The discrepancy between the actual number of facets and McMullen's upper bound is particularly stark in the unconstrained Example 1 (i.e. the pure object-allocation setting of BM) with n objects and n agents. The polytope here has more extreme points than any polytope arising for any other set of outcome constraints. The large number of extreme points implies a larger McMullen upper bound. Yet, the polytope has only $n(n+2)$ facets:

n^2 facets corresponding to the non-negativity constraints, one for each coordinate, and $2n$ facets for the row and column bistochasticity constraints. See Billera and Björner (2018) for more related results on the number of facets of some related polytopes.

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