A behavioral characterization of the Drift Diffusion Model and its multi-alternative extension to choice under time pressure

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December 25, 2018

Abstract

In this paper, we provide an axiomatic foundation for the value-based version of the Drift Diffusion Model (DDM) of Ratcliff (1978), a successful model that describes two-alternative speeded decisions between (fast-moving) consumer goods.

Our axioms present a test for model misspecification and connect the externally observable properties of choice with an established neurophysiological account of how choice is internally implemented.

We then extend our axiomatic analysis to multi-alternative choice under time pressure.

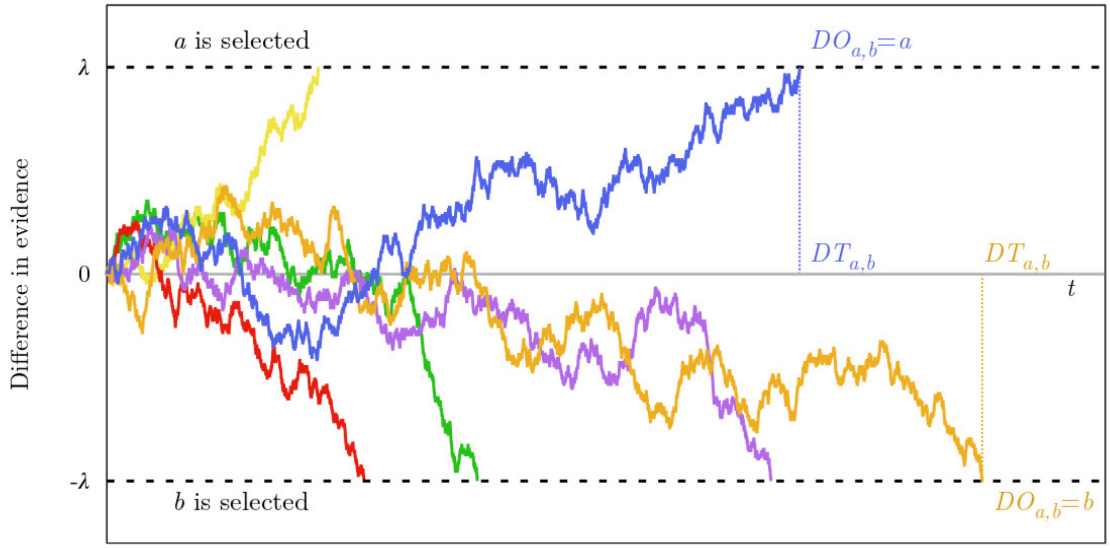
A successful model to describe two-alternative speeded decisions between (fast-moving) consumer goods is the value-based version of the Drift Diffusion Model (DDM) of Ratcliff [41] proposed by Milosavljevic et al. [34] and neurophysiologically motivated by Shadlen and Shohamy [51] in terms of sequential sampling from memory. Here we provide an axiomatic foundation for this model and a simple way to elicit its parameters from behavioral data.

When eye-tracking data are also available, our characterization allows to test the Metropolis-DDM algorithm, a recent multi-alternative extension of the DDM due to Cerreia-Vioglio et al. [6], and to identify its parameters.

The DDM is by now a paradigm for choice between pairs $\{a, b\}$ of alternatives. It explains a wide range of behavioral and neuroscientific data, it has a compelling neurophysiological interpretation, and it is optimal in terms of sequential sampling.[1] Recently, it has also been shown to successfully describe a wide range of purchasing decisions (from snacks to consumer electronics, to household items).[2] This neuro-computational algorithm assumes that decisions are made by accumulating noisy information about the two alternatives, a and b, over time until the net evidence in favor of one exceeds a prespecified threshold, say λ, at which time the favored alternative is selected. The presence of noise in the accumulation of information implies that choice between the same pair of alternatives does not always terminate at the same time and does not always lead to the same outcome. More formally, the DDM describes how linear evidence accumulation with white Gaussian noise generates the random variables *decision time*, $\mathrm{DT}_{a,b}$, and *decision outcome*, $\mathrm{DO}_{a,b}$, for choice in a two-alternative set $\{a, b\}$.

[1] See, e.g., Bogacz et al. [2], Hare et al. [19], Ratcliff et al. [42] and [43], and Fudenberg et al. [16].

[2] See, e.g., Roe et al. [48], Milosavljevic et al. [34], Krajbich et al. [24], and Clithero [8].



Moreover, the DDM naturally captures speed-accuracy tradeoffs: lower thresholds λ produce faster but less accurate responses, whereas higher thresholds λ produce more accurate but slower responses. This feature is particularly relevant for choice under time pressure: empirical evidence confirms the intuition that higher pressure induces lower thresholds.[3]

Our main contribution is providing necessary and sufficient conditions on the observables – that is, on choice frequencies and decision times – that guarantee that the agent behaves "as if" implementing the DDM. In the tradition of psychophysics, these conditions are called "axioms" and can be seen both as an empirical test of the model and as a measurement tool for its parameters.[4] The twist of our approach is combining the choice frequency and decision time components into an axiomatic characterization. Both these observables are at the heart of most psychophysical theories (see, e.g., Luce [26] and [27]). Yet, while the former has been studied in great axiomatic detail under the name of "random choice", the latter has not been analyzed from this perspective, with the exception of the recent Echenique and Saito [11]. We also extend our results to the popular variation of the DDM of Wagenmakers et al. [55], called EZ-DDM, that allows to incorporate the non-deliberative part of response times.

Beyond falsifiability of the DDM theory and a better understanding of its behavioral implications, an important experimental advantage of our main representation theorem is that it does not require a parameter fitting routine, but allows to elicit the agent's utility function and decision threshold directly from behavioral data. Also, our framework can be immediately applied to common experimental setups in which each participant contributes only a moderate amount of data and the error rate is low – see Wagenmakers et al. [55], Lerche and Voss [25], and van Ravenzwaaij et al. [44].

Last but not least, our axiomatization extends beyond two-alternative choice modelling. Indeed, we show that it permits to test and to elicit the parameters of the multi-alternative choice procedure under time pressure of Cerreia-Vioglio et al. [6]. In so doing, we also generalize their Metropolis-DDM algorithm to allow for the formation of consideration sets. Because of the importance of these sets in economics and marketing,[5] this generalization is another salient contribution of the present paper.

[3] See, e.g., Milosavljevic et al. [34], Karsilar et al. [21], and the discussion in Ortega and Stocker [38].

[4] Classical references are Luce [26] and Luce and Suppes [28].

[5] See, e.g., Shocker et al. [52], Roberts and Nedungadi [47], Peter and Olson [39], Eliaz and Spiegler [12], Masatlioglu et al. [32], Hauser [20], Manzini and Mariotti [31], and Gaynor et al. [17].

More about the related literature The literature on the DDM is vast but non-axiomatic. We refer readers to the reviews of Fehr and Rangel [15] and Ratcliff et al. [43]. Webb [56] studies the relation between the DDM (and bounded accumulation models, in general) and random utility models. The optimality of the DDM in terms of sequential sampling is analyzed by Gold and Shadlen [18] and Bogacz et al. [2] in the classical case, and by Fudenberg et al. [16] and Epstein and Ji [13] in the Bayesian case.

The extension of the DDM to menus of $N > 2$ alternatives is a non-trivial issue and different generalizations, with significantly different behavioral and neurobiological properties, have been proposed. See, e.g., Roe et al. [48], Usher and McClelland [53], Anderson et al. [1], McMillen and Holmes [33], Bogacz et al. [3], Ditterich [9], Krajbich and Rangel [23], and Reutskaja et al. [45]. In most of these models, the choice task is assumed to **simultaneously activate N accumulators**, each of which is primarily sensitive to one of the alternatives and integrates evidence relative to that alternative. Choices are then made based on absolute or relative evidence levels. In contrast, the Metropolis-DDM algorithm of Cerreia-Vioglio et al. [6] builds on **sequential activation of 2 accumulators** and Markovian exploration of the menu of alternatives. This feature makes the Metropolis-DDM algorithm more realistic in view of both the available eye-tracking evidence – see, e.g., Russo and Rosen [50] and Russo and Leclerc [49] – and of the known limitations of working memory – see, e.g., Luck and Vogel [29] and Vogel and Machizawa [54]. The same feature allows for model-testing and permits parameters' elicitation by analyzing binary comparisons only. These comparisons are the most studied in many fields of decision theory and their quantitative and experimental analysis is consequently well developed.

1 The DDM for value-based decisions

Let A be a choice set consisting of at least three distinct alternatives. The DDM is a model of binary comparison between pairs of alternatives a and b in A. According to this model, noisy evidence about the alternatives is accumulated until it reaches some threshold $\lambda > 0$, at which point a decision is taken. Specifically, either alternative is selected as soon as the net evidence in its favor attains level λ. In a neurophysiological perspective, the comparison of a and b is assumed to activate two neuronal populations whose activities (firing rates) provide evidence for the two alternatives. Denote their mean activities by $u(a)$ and $u(b)$, and assume that each experiences instantaneous independent white noise fluctuations modeled by uncorrelated Wiener processes W_a and W_b. Evidence accumulation in favor of a and b is then represented by the two Brownian motions with drift $V_a(t) \equiv u(a)t + \sigma W_a(t)$ and $V_b(t) \equiv u(b)t + \sigma W_b(t)$, respectively.[6] With this,

- the *net evidence* in favor of a against b is given, at each $t \in (0, \infty)$, by the difference

$$Z_{a,b}(t) \equiv V_a(t) - V_b(t) = [u(a) - u(b)]\, t + \sigma\sqrt{2}W(t)$$

  where W is the Wiener process $(W_a - W_b)/\sqrt{2}$;

- the comparison ends when $Z_{a,b}(t)$ reaches either the barrier λ or $-\lambda$; so, the *decision time* is the random variable

$$\mathrm{DT}_{a,b} \equiv \min\{t : Z_{a,b}(t) = \lambda \text{ or } Z_{a,b}(t) = -\lambda\}$$

[6] See, e.g., Bogacz et al. [2] and Fudenberg et al. [16].

- when the comparison ends (at random time $\mathrm{DT}_{a,b}$), the agent selects a if the upper barrier λ has been reached, and selects b otherwise (if the lower barrier $-\lambda$ has been reached); so, the *decision outcome* is the random variable

$$\mathrm{DO}_{a,b} \equiv \begin{cases} a & \text{if } Z_{a,b}\left(\mathrm{DT}_{a,b}\right) = \lambda \\ b & \text{if } Z_{a,b}\left(\mathrm{DT}_{a,b}\right) = -\lambda \end{cases}$$

The probability of choosing a from $\{a, b\}$ is thus

$$P_{a,b} \equiv \mathbb{P}\left[\mathrm{DO}_{a,b} = a\right]$$

and its explicit logistic formula

$$P_{a,b} = \frac{1}{1 + e^{-\frac{\lambda}{\sigma^2}[u(a)-u(b)]}} \tag{1}$$

can be already found in Ratcliff [41]. In particular, the choice of an inferior alternative, a if $u(a) \leq u(b)$ or b if $u(b) \leq u(a)$, is called an error. Its probability is the *error rate*

$$\mathrm{ER}_{a,b} \equiv \min\left\{P_{a,b}, P_{b,a}\right\} = \frac{1}{1 + e^{\frac{\lambda}{\sigma^2}|u(a)-u(b)|}}$$

The explicit formulas of the distribution of $\mathrm{DT}_{a,b}$ and of its moments are also well known (see the appendix). For example, its mean is

$$\mathrm{MDT}_{a,b} \equiv \mathbb{E}\left[\mathrm{DT}_{a,b}\right] = \frac{\lambda}{u(a) - u(b)} \tanh \frac{\lambda\left[u(a) - u(b)\right]}{2\sigma^2} \tag{2}$$

As intuitive, the latter increases with the amount of net evidence required to decide, as well as with the payoff proximity of the alternatives.[7]

Normalizations Notice that the parameters u, σ, and λ are defined up to a positive scalar multiple. If *all* of them are multiplied by a constant $\alpha > 0$, the predictions of the DDM are unchanged. For instance, choosing $\alpha = 1/\sigma$ amounts to normalize the noise σ of the Brownian motions $\{V_a\}_{a \in A}$ and to replace σ with 1, u with $\hat{u} = u/\sigma$, and λ with $\hat{\lambda} = \lambda/\sigma$.[8]

A different normalization, typical of the mathematical psychology literature, consists in setting $\lambda = 1$. It obviously corresponds to $\alpha = 1/\lambda$.

Finally, observe that u is actually cardinally unique, that is, defined up to positive scalar multiplication as well as to translation by an additive constant. For instance, in a neurophysiological perspective, it may be desirable to normalize the range of u to $[0, 100]$. This requires u to be bounded (and nonconstant) and leads to the transformation

$$u \mapsto \frac{100}{\sup_{a \in A} u(a) - \inf_{a \in A} u(a)} \left(u - \inf_{a \in A} u(a)\right) = \alpha \left(u - \inf_{a \in A} u(a)\right)$$

[7] In some cases the alternatives a and b may play different roles, say b is the *status quo* or the *incumbent solution* of a decision problem. The amount of net evidence required to maintain b, call it β, may then be different from the amount of net evidence λ required to switch to a. In these cases, it is necessary to replace $-\lambda$ with $-\beta$ in the expressions of $\mathrm{DT}_{a,b}$ and $\mathrm{DO}_{a,b}$, and expressions (1) and (2) should be modified accordingly. Our multi-alternative generalization of the DDM can be extended to these cases. See the appendix.

[8] This is the normalization that we will adopt, when we do not maintain the generic expression. Another normalization of the noise coefficient, popular in behavioral experiments, corresponds to $\sigma\sqrt{2} = 0.1$ and it determines a choice of $\alpha = \sqrt{2}/20\sigma$.

2 Observability and measurement

In terms of external observability of the DDM, assume that the analyst can observe the agent's choices between a and b several times. These observations produce an empirical *choice frequency* $P^o_{a,b} = 1 - P^o_{b,a}$ and an empirical *mean decision time* $\mathrm{MDT}^o_{a,b} = \mathrm{MDT}^o_{b,a}$. Formally, we call *observables* any pair of square matrices

$$\left(P^o, \mathrm{MDT}^o\right) = \left\{\left(P^o_{a,b}, \mathrm{MDT}^o_{a,b}\right) : P^o_{a,b} + P^o_{b,a} = 1 \text{ and } \mathrm{MDT}^o_{a,b} = \mathrm{MDT}^o_{b,a} \text{ for all } a \neq b\right\}$$

Example 1 *With three alternatives, the matrices are*

$$\begin{bmatrix} * & P^o_{a,b} & P^o_{a,c} \\ P^o_{b,a} & * & P^o_{b,c} \\ P^o_{c,a} & P^o_{c,b} & * \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} * & \mathrm{MDT}^o_{a,b} & \mathrm{MDT}^o_{a,c} \\ \mathrm{MDT}^o_{b,a} & * & \mathrm{MDT}^o_{b,c} \\ \mathrm{MDT}^o_{c,a} & \mathrm{MDT}^o_{c,b} & * \end{bmatrix}$$

The elements on the diagonal, which are conceptually meaningless, can be arbitrarily specified.

Here the superscript "o" stands for "observable" (or "observed"). For instance, $\mathrm{ER}^o_{a,b} \equiv \min\left\{P^o_{a,b}, P^o_{b,a}\right\}$ is the *observed (experimental) error rate*, while $\mathrm{ER}_{a,b} \equiv \min\left\{P_{a,b}, P_{b,a}\right\}$ is the *DDM (theoretical) error rate.*

Next we show that the DDM is characterized by simple verifiable conditions on observables that we state as axioms. They are required to hold for all distinct $a, b, c \in A$.

Axiom 1 (Positivity) $P^o_{a,b} > 0$ *and* $\mathrm{MDT}^o_{a,b} > 0$.

This axiom requires that choice be a genuinely stochastic and time consuming process. It dates back to Luce [26].

Axiom 2 (Product rule) $P^o_{a,b} P^o_{b,c} P^o_{c,a} = P^o_{a,c} P^o_{c,b} P^o_{b,a}$.

The product rule asserts that choice cycles

$$a \to c \to b \to a \qquad \text{and} \qquad a \to b \to c \to a$$

must be observed with the same probability. In other words, violations of transitivity are only due to noise (mistakes). Luce and Suppes [28, p. 341] show that, together with positivity, this axiom characterizes the Luce model of binary choice.[9] In turn, via the relation

$$P^o_{a,b} \equiv F\left(v\left(a\right) - v\left(b\right)\right)$$

this model allows to elicit, from the observable choice frequency $P^o_{a,b}$, two unobservable objects of interest for the psychometric analyst: a *Fechnerian value* v, used to measure the difference in final intensity of "subjective sensations" generated by the stimuli a and b, and a *discrimination function* F, which describes how this difference affects discrimination. Specifically, in the model of Luce the discrimination function is logistic

$$F\left(s\right) = \frac{1}{1 + e^{-s}} \qquad \forall s \in \mathbb{R}$$

[9] See Luce [26, Ch. 1-2]. This axiom is often expressed in terms of odds (it coincides with axiom EZ2 of the next section).

and the difference $v(a) - v(b)$ can be retrieved by inverting F; so that

$$v(a) - v(b) = \operatorname{logit} P^o_{a,b} = \log \frac{P^o_{a,b}}{1 - P^o_{a,b}} \qquad \forall a, b \in A$$

is given by the logarithm of the observed odds for a against b.[10]

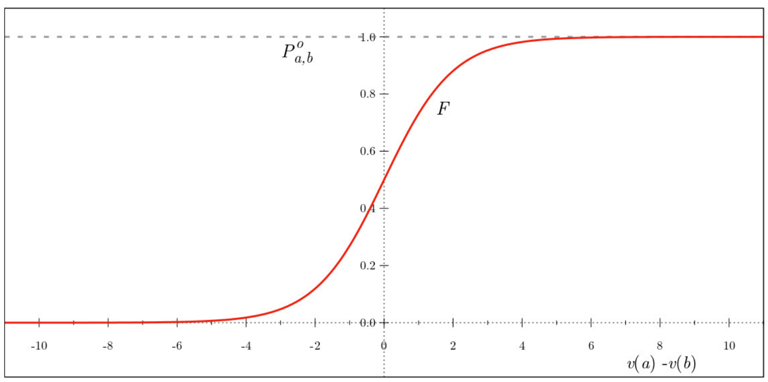
As a consequence, the *error rate* is

$$\mathrm{ER}^o_{a,b} = \frac{1}{1 + e^{|v(a) - v(b)|}} \qquad \forall a, b \in A$$

and the absolute intensity difference $|v(a) - v(b)|$, representing the *ease of comparison*,[11] is given by the log-odds of a correct response

$$|v(a) - v(b)| = \operatorname{logit}\left(1 - \mathrm{ER}^o_{a,b}\right) = \log \frac{1 - \mathrm{ER}^o_{a,b}}{\mathrm{ER}^o_{a,b}}$$

All these considerations are made clear by a plot of the discrimination function F



We are now ready to state our final axiom.

Axiom 3 (Invariance) $\mathrm{MDT}^o_{a,b} \dfrac{\operatorname{logit}\left(1 - \mathrm{ER}^o_{a,b}\right)}{1/2 - \mathrm{ER}^o_{a,b}} = \mathrm{MDT}^o_{a,c} \dfrac{\operatorname{logit}\left(1 - \mathrm{ER}^o_{a,c}\right)}{1/2 - \mathrm{ER}^o_{a,c}}$.

To interpret, observe that $\mathrm{ER}^o_{a,b}$ always ranges in $[0, 1/2]$, so $1/2 - \mathrm{ER}^o_{a,b}$ measures the *accuracy of comparison*, and that the invariance axiom requires the existence of a constant $\kappa > 0$ for which

$$\mathrm{MDT}^o_{a,b} = \kappa \cdot \frac{1/2 - \mathrm{ER}^o_{a,b}}{\operatorname{logit}\left(1 - \mathrm{ER}^o_{a,b}\right)} \qquad \forall a, b \in A$$

In words, this formula thus says that "mean decision time is proportional to the desired accuracy and inversely proportional to ease of comparison".

The next theorem, our first main contribution, shows that observables can be explained by the DDM if and only if they satisfy the previous axioms.

[10] As detailed in (4), the *odds* of an event are the ratio of the probability of the event itself to the probability of its complement. Note that, when we write "$\forall a, b \in A$" we intend "for all distinct a and b in A".

[11] Indeed, the higher this absolute value, the easier the discrimination between a and b (see also Fudenberg et al. [16]).

Theorem 1 *Let* (P^o, MDT^o) *be the observables. The following are equivalent:*

(i) P^o *and* MDT^o *satisfy positivity, the product rule, and invariance;*

(ii) there exist a function $u : A \to \mathbb{R}$ *and two coefficients* $\sigma > 0$ *and* $\lambda > 0$ *such that* $P^o = P$ *and* $\mathrm{MDT}^o = \mathrm{MDT}$.

In this case, $\hat{\lambda} = \lambda/\sigma$ *is unique and* $\hat{u} = u/\sigma$ *is unique up to an additive constant. In particular,*

$$\hat{\lambda} = \sqrt{\mathrm{MDT}^o_{a,b} \frac{\operatorname{logit}\left(1 - \mathrm{ER}^o_{a,b}\right)}{1 - 2\mathrm{ER}^o_{a,b}}} \qquad \text{and} \qquad \hat{u}(a) - \hat{u}(b) = \frac{\operatorname{logit} P^o_{a,b}}{\hat{\lambda}} \tag{3}$$

for all $a \neq b$ *in* A.

This theorem has several noteworthy consequences. First, the identification of $\hat{\lambda}$ and $\hat{u}$ allows for inter-agent and intra-agent comparative statics. For instance, it permits to say that agent 1 is "more reflective" than agent 2 if and only if $\hat{\lambda}_1 > \hat{\lambda}_2$,[12] or to say that agent 1 is "more risk averse" than agent 2 if and only if the certainty equivalents corresponding to $\hat{u}_1$ are smaller than those corresponding to $\hat{u}_2$.[13]

Second, this theorem guarantees that the function u is cardinally unique and that, given u, the coefficients σ and λ are both unique. Specifically, if instead of u we consider $\alpha u + \beta$, with $\alpha > 0$ and $\beta \in \mathbb{R}$, the diffusion coefficient σ and the threshold λ must be both multiplied by α itself.[14]

Finally, Theorem 1 shows that utility differences are cardinally measured jointly by choice probabilities and decision times. In this regard, the next proposition shows that alone either choice probabilities or decision times are sufficient to ordinally measure such differences.[15]

Proposition 2 *Given a function* $u : A \to \mathbb{R}$ *and two coefficients* $\sigma > 0$ *and* $\lambda > 0$, *if* $a \neq b$ *and* $a' \neq b'$ *belong to* A, *then the following are equivalent:*

(i) $|u(a) - u(b)| \leq |u(a') - u(b')|$;

(ii) $\mathrm{ER}_{a,b} \geq \mathrm{ER}_{a',b'}$;

(iii) $\mathrm{MDT}_{a,b} \geq \mathrm{MDT}_{a',b'}$;

(iv) $\mathrm{DT}_{a,b}$ *stochastically dominates* $\mathrm{DT}_{a',b'}$.

Moreover, $\mathrm{DT}_{a,b}$ *and* $\mathrm{DO}_{a,b}$ *are independent random variables.*

The mathematical novelty of this proposition is the equivalence of point (iv) with the remaining points (i), (ii), and (iii). The equivalence of these three points highlights a significant feature of the value-based DDM: two pairs of alternatives present the same absolute difference in intensity of stimuli if, and only if, they generate the same discrimination error if, and only if, their discrimination time is on average the same. This means that, under the DDM assumptions, the measurement of these differences either by error rates à la Fechner – see, e.g., Luce [26, Ch. 2] and Falmagne [14, Ch. 4] – or by decision times à la Cattel [5] actually coincide. In this way, two of the historically most important hypotheses of classical psychophysics are reconciled.

[12] Even when they have different utility functions $\hat{u}_1$ and $\hat{u}_2$, but provided they are choosing in the same conditions.

[13] Even when they have different thresholds $\hat{\lambda}_1$ and $\hat{\lambda}_2$, but provided choice between lotteries is observed.

[14] Recall our previous discussion on normalizations.

[15] See Echenique and Saito [11] for a general revealed-preference approach to ordinal measurement of utility differences through response times.

3 A useful variation: the EZ-DDM

In behavioral experiments it is often difficult to observe decision time in an accurate manner. What is often observed is a *response time* which consists of a sum

$$\mathrm{RT}^o_{a,b} \equiv T^h_{a,b} + \mathrm{DT}^h_{a,b}$$

where $\mathrm{DT}^h_{a,b}$ is the actual decision time (the superscript "h" stands for "hidden") and $T^h_{a,b}$ is a fixed amount of time that in every trial takes place before the initiation of the comparison process proper (e.g., if b is the status quo, the time needed to select alternative a for comparison among all the elements of $A \setminus \{b\}$) and after the conclusion of the process (e.g., the time that it takes to execute the motor commands necessary to implement the resulting choice).

This leads to consider an augmented DDM, called EZ-DDM (Wagenmakers et al. [55]) or *simple DDM* (Bogacz et al. [2] and Milosavljevic et al. [34]), in which decision time is replaced by *response time*

$$\mathrm{RT}_{a,b} \equiv T_{a,b} + \mathrm{DT}_{a,b}$$

The quasi-positive symmetric matrix $T = [T_{a,b}]_{a,b\in A}$ is called *latency matrix.*[16]

In this section, we combine the intuitions of the previous section and those of Wagenmakers et al. [55] to provide an axiomatization of the EZ-DDM.

We call *augmented observables* any triplet of square matrices

$$\begin{aligned}(P^o, \mathrm{MRT}^o, \mathrm{VRT}^o) \quad = \quad & \big\{ \big(P^o_{a,b}, \mathrm{MRT}^o_{a,b}, \mathrm{VRT}^o_{a,b}\big) : P^o_{a,b} + P^o_{b,a} = 1 \text{ and } \mathrm{MRT}^o_{a,b} = \mathrm{MRT}^o_{b,a} \\ & \qquad \text{and } \mathrm{VRT}^o_{a,b} = \mathrm{VRT}^o_{b,a} \text{ for all}\, a \neq b \big\}\end{aligned}$$

Here P^o is the matrix of empirical choice frequencies as in the previous section, MRT^o and VRT^o are the symmetric matrices of empirical *mean response times* and empirical *variances of response times*, respectively. The use of these observable quantities dates back to the mentioned Cattel [5].

To ease notation, we denote the observed *odds* for a against b by

$$R^o_{a,b} \equiv \frac{P^o_{a,b}}{P^o_{b,a}} = \frac{\text{No. of times } a \text{ is chosen from } \{a,b\}}{\text{No. of times } b \text{ is chosen from } \{a,b\}} \tag{4}$$

The following axioms are required to hold for all distinct $a, b, c \in A$.

Axiom EZ1 $R^o_{a,b} > 0$, $\mathrm{MRT}^o_{a,b} > 0$, *and* $\mathrm{VRT}^o_{a,b} > 0$.

Axiom EZ2 $R^o_{a,b} = R^o_{a,c} R^o_{c,b}$.

Axiom EZ3 $\mathrm{VRT}^o_{a,b} \dfrac{\left(R^o_{a,b}+1\right)^2 \left(\log R^o_{a,b}\right)^3}{\left(R^o_{a,b}\right)^2 - 2R^o_{a,b}\log R^o_{a,b} - 1} = \mathrm{VRT}^o_{a,c} \dfrac{\left(R^o_{a,c}+1\right)^2 \left(\log R^o_{a,c}\right)^3}{\left(R^o_{a,c}\right)^2 - 2R^o_{a,c}\log R^o_{a,c} - 1}$.

Axiom EZ4 $2\left(\mathrm{MRT}^o_{a,b}\right)^2 \geq \mathrm{VRT}^o_{a,b} \dfrac{\left(R^o_{a,b}-1\right)^2 \log R^o_{a,b}}{\left(R^o_{a,b}\right)^2 - 2R^o_{a,b}\log R^o_{a,b} - 1}$.

Axioms EZ1 and EZ2 correspond to the positivity and product rule axioms introduced in the previous section, while axioms EZ3 and EZ4 are technical and inspired by the non-axiomatic analysis of Wagenmakers et al. [55].

[16] A matrix is quasi-positive if, and only if, its off-diagonal terms are all positive.

Theorem 3 *Let* $(P^o, \mathrm{MRT}^o, \mathrm{VRT}^o)$ *be the augmented observables. The following are equivalent:*

(i) P^o*,* MRT^o*, and* VRT^o *satisfy EZ1, EZ2, EZ3, and EZ4;*

(ii) there exist a function $u : A \to \mathbb{R}$*, two coefficients* $\sigma > 0$ *and* $\lambda > 0$*, and a symmetric quasi-positive* $A \times A$ *matrix* T *such that* $P^o = P$*,* $\mathrm{MRT}^o = \mathrm{MRT}$*, and* $\mathrm{VRT}^o = \mathrm{VRT}$*.*

In this case, $\hat{\lambda} = \lambda/\sigma$ *is unique, and* $\hat{u} = u/\sigma$ *is unique up to an additive constant. In particular,*

$$\hat{\lambda} = \sqrt[4]{\mathrm{VRT}^o_{a,b} \frac{\left(R^o_{a,b}+1\right)^2 \left(\log R^o_{a,b}\right)^3}{2\left(R^o_{a,b}\right)^2 - 4R^o_{a,b}\log R^o_{a,b} - 2}} \qquad \text{and} \qquad \hat{u}(a) - \hat{u}(b) = \frac{\log R^o_{a,b}}{\hat{\lambda}}$$

for all $a \neq b$ *in* A*. Moreover,* T *is unique and given by*

$$T_{a,b} = \mathrm{MRT}^o_{a,b} - \sqrt{\mathrm{VRT}^o_{a,b} \frac{\left(R^o_{a,b}-1\right)^2 \log R^o_{a,b}}{2\left(R^o_{a,b}\right)^2 - 4R^o_{a,b}\log R^o_{a,b} - 2}}$$

for all $a \neq b$ *in* A*.*

Clearly, the DDM corresponds to the case in which $T_{a,b} = 0$ for all $a \neq b$. Thus, Theorem 3 provides an alternative characterization of the DDM based on response times' means and variances.[17] Hence, we write $\mathrm{DM}(\hat{u}, \hat{\lambda}, T)$ to denote both the DDM and the EZ-DDM (with $\hat{\sigma} = 1$).

4 An application: the Metropolis-DDM algorithm

In this final section, we present an application of the previous analysis to multi-alternative choice under time pressure. Here A represents the set of **available alternatives** and an **exogenous time limit** t is imposed on the agent. For example, they might have to choose one of the following 9 available snacks in 4 seconds:

Our analysis of this problem is based on the Metropolis-DDM algorithm of Cerreia-Vioglio et al. [6]. Although the section is self contained, we refer the reader to [6] for an in-depth discussion of the algorithm and of its relations with the literature.[18] The novel contributions of

[17] Notice that the axiom delivering the DDM is obtained by replacing the inequality in EZ4 with an equality. In the proof of Theorem 3, we discuss an additional axiom that makes $T_{a,b}$ constant, that is, independent of a, b (see Footnote 32).

[18] The same authors are presently conducting experimental tests of the algorithm itself.

the present section consist, first, in showing how the axioms we introduced so far allow to study multi-alternative choice environments and, second, in generalizing the original Metropolis-DDM algorithm to allow for the formation of consideration sets.[19] For example, our agent might restrict his attention to the subset C of available sweet snacks:

Formally, given a set A of available alternatives, a *consideration set* is a subset C of A consisting of the items among which a consumer actually chooses in a given decision episode. These sets are central in marketing,[20] where their formation is assumed to be the first step in a two-step choice process (the second step of which consists in choosing an alternative from the consideration set). For this reason, as Ringel and Skiera [46] write, they are "the ultimate arbiters of the competition" among brand managers, whose objective is to maximize the chances that their products belong to these sets.[21]

Before describing the Metropolis-DDM algorithm, we recall some eye-tracking experimental findings (in italics) on multi-alternative choice under time pressure that inspired it,[22] along with (in roman) the corresponding "ingredient" of the algorithm itself.

F1 *Multi-alternative choice procedures are composed primarily of sequential pairwise comparisons, in which actual evaluative processing takes place.*

We describe these pairwise comparisons via the Drift Diffusion Model DM (u, λ, T).

F2 *Increases in time pressure lead to acceleration of information processing, often at the cost of accuracy.*

We allow the threshold λ and the latency matrix T to depend on the deadline t.

F3 *Search strategies and consideration sets are adapted to time constraints and affected by visual saliency, and agents do not eliminate alternatives after they are rejected in a previous pairwise comparison.*

---

[19] We also extend [6] by considering pairwise EZ-DDM comparisons (instead of pure DDM ones). But we maintain the algorithm name unchanged.

[20] See the review of Shocker et al. [52], Roberts and Nedungadi [47] where an issue of the International Journal of Research in Marketing on this topic is foreworded, and the more recent Hauser [20], or Peter and Olson [39] for a textbook treatment.

[21] More recently, consideration sets have also attracted attention in economics. See, e.g., Eliaz and Spiegler [12], Masatlioglu et al. [32], Manzini and Mariotti [31], and Gaynor et al. [17].

[22] See Russo and Rosen [50], Russo and Leclerc [49], Nowlis [37], Pieters and Warlop [40], Chandon et al. [7], Krajbich et al. [22], Krajbich and Rangel [23], Reutskaja et al. [45], Milosavljevic et al. [35], and Karsilar et al. [21].

We describe consideration sets by a partition $\mathcal{C}$ of A – for example, $\mathcal{C} = \{$"sweet snacks", "salty snacks"$\}$ – that is permitted to depend on the time constraint t. Moreover, we denote by

$$Q(a \mid b)$$

the probability of considering a new alternative a for comparison with the temporary solution b. This probability is allowed to vary with t too.

F4 *Agents' exploration of menus is driven by the similarity and proximity of available alternatives, that is, on their perceptual distance.*

This finding suggests a simple parametric form for Q that, although not necessary for our analysis, is intuitive and performs well in simulations:

$$Q(a \mid b) = \frac{1}{|A| - 1} \cdot \frac{1}{d(a,b)^{\gamma}} \qquad \forall a, b \in A$$

Here d is a perceptual distance (that is, a symmetric function) between alternatives such that $\min_{a \neq b} d(a,b) = 1$ and $\max_{a \neq b} d(a,b) \leq \infty$; it captures both physical proximity and subjective similarity. While $\gamma \in (0, \infty)$ is an exploration aversion parameter. When γ is very large, the agent basically regards as close only the nearest neighbors of the temporary solution b; instead, when γ is very small, all the considered alternatives are essentially equally distant. For example, in the case of our 9 snacks, a simple perceptual distance is given by

$$d(a,b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \text{ and } b \text{ are adjacent and are either both sweet or both salty} \\ 2 & \text{if } a \text{ and } b \text{ are not adjacent and are either both sweet or both salty} \\ \infty & \text{if one is sweet and the other salty} \end{cases}$$

Since this distance takes into account the sweet/salty partition, so does the corresponding transition probability matrix Q.

F5 *The initial fixation is random and independent of value.*[23]

Our final ingredient is thus an initial probability distribution μ on A, also this distribution may depend on t.

Together, all our ingredients suggest the following decision procedure. When a menu A and a deadline t are given, our agent first selects a sub-menu C of A and an initial element b in C according to the consideration sets' partition $\mathcal{C}$ and the initial distribution μ. Then, they consider an alternative solution a in C with probability $Q(a \mid b)$, and compares it to b via $\mathrm{DM}(u, \lambda, T)$. If proposal a is judged superior to incumbent b, then a becomes the new incumbent and another proposal c in C is considered and compared to a via $\mathrm{DM}(u, \lambda, T)$; otherwise, b maintains its incumbent status and another proposal is considered and compared. This sequential exploration and comparison continues until the time t available to decide expires and the incumbent solution is chosen from the consideration set C.

Before describing formally this decision procedure, a couple of remarks are in order. First, it is important to observe that **the axioms of the previous sections**, together with the eye-tracking detection of binary comparisons, make these assumptions testable and their parameters

[23] But possibly dependent on consideration sets and visual saliency.

quantifiable. So, it is the analysis of the first part of this paper that makes **empirically relevant** what we propose here.

Second, note that the nature of consideration sets we propose is both set-theoretic and probabilistic. Intuitively, a partition $\mathcal{C}$ of A consists of consideration sets if once a set $C \in \mathcal{C}$ is selected by the agent, then:

1. any element of C can be considered (with strictly positive probability),
2. no element outside C can be considered.

Now, if the agent explores alternatives according to transition matrix Q, this means that given any C in $\mathcal{C}$ and any item c in C:

1. it is possible to reach from c any element inside C in a finite number of transitions,
2. it is impossible to reach from c any element outside C in a finite number of transitions.

In sum, the partition $\mathcal{C}$ must coincide with the partition of communicating classes determined by the exploration matrix Q.[24]

We are now ready to present our multi-alternative choice model, a generalization of the Metropolis-DDM algorithm of Cerreia-Vioglio et al. [6].

Metropolis-DDM Algorithm

Input: Given $t > 0$, set $\mu = \mu_t$, $Q = Q_t$, $\lambda = \lambda_t$, and $T = T_t$.

Start: Draw a from A according to μ:

- set $s_0 = 0$,
- set $b_0 = a$.

Repeat: Draw a from A according to $Q(\cdot \mid b_n)$ and compare it to b_n via $\mathrm{DM}(u, \lambda, T)$:

- set $s_{n+1} = s_n + \mathrm{RT}_{a,b_n}$,
- set $b_{n+1} = \mathrm{DO}_{a,b_n}$,

until $s_{n+1} > t$.

Stop: Set $b^* = b_n$.

Output: Choose b^* from A.

---

[24] Specifically, given any $c \in C$, if $a \notin C$, there is no finite sequence $c = a_0, a_1, ..., a_n = a$ such that $\prod_{k=0}^{n-1} Q(a_{k+1} \mid a_k) > 0$ – in particular, $Q(a \mid c) = 0$ for all $a \notin C$ In contrast, if $a \in C$, such a finite sequence exists – in particular, $a_0, a_1, ..., a_n \in C$.

This algorithm can be seen as a parsimonious variation of the standard optimal search algorithm that takes into account the presence of time pressure. In the standard algorithm, the agent begins by selecting an initial element b in A, then at each iteration they compare an incumbent and a proposal, and discards permanently the rejected alternative, until the menu is exhausted. Here, the presence of a deadline may lead to the formation of consideration sets, and the possibility of mistakes makes it inadvisable to eliminate proposals that have been rejected in a previous comparison. Nonetheless, the sequential "explore-and-compare" logic of the two procedures is similar.

By implementing the Metropolis-DDM algorithm, the probability of selecting a given incumbent b is

$$M_t\left(a \mid b\right) = Q_t\left(a \mid b\right) P_{a,b} \qquad \forall a, b \in A$$

The transition probability $M_t\left(a \mid b\right)$ combines the stochasticity of the proposal mechanism $Q_t(a \mid b)$ and that of the acceptance/rejection rule $P_{a,b}$ (which also depends on t via λ_t). Therefore, after n iterations of the repeat-until loop, the probability of b being the incumbent is the b-th component of the row vector $\mu_t M_t^n$. The next result shows that the limiting behavior of this sequence turns out to be classical softmaximization, conditional on the communicating classes determined by Q.

Theorem 4 *Let $u : A \to \mathbb{R}$ be a function, $\lambda_t > 0$ a coefficient, and Q_t a symmetric stochastic $A \times A$ matrix. Then, M_t is reversible with respect to the multinomial logit distribution*

$$p_A^{(u,\lambda_t)}\left(a\right) = \frac{e^{\lambda_t u(a)}}{\sum_{b \in A} e^{\lambda_t u(b)}} \qquad \forall a \in A$$

and, given any probability distribution μ_t on A,

$$\lim_{n \to \infty} \mu_t M_t^n = \sum_{C \in \mathcal{C}_t} \mu_t\left(C\right) p_C^{(u,\lambda_t)}$$

where $\mathcal{C}_t$ is the partition of A into its communicating classes with respect to Q_t.[25]

In particular, if Q_t is irreducible, then M_t is irreducible, aperiodic, and $\lim_{n \to \infty} \mu_t M_t^n = p_A^{(u,\lambda_t)}$.

If Q_t is irreducible,[26] the Metropolis-DDM algorithm thus approximates the *multinomial logit* (or *softmax*) distribution $p_A^{(u,\lambda_t)}$, irrespective of the initial distribution μ_t. Otherwise, the algorithm selects a consideration sub-menu C of A and approximates the conditional multinomial logit there.

Beyond the mathematical novelty, the conceptual innovation of the algorithm presented above relative to the original Metropolis-DDM is allowing the exploration strategy – in particular, the consideration sets' structure – to depend on the time constraint. For instance, this is potentially relevant in today's marketplace in which web-based stores offer consumers immense choice sets and life trends dramatically reduce deliberation times.

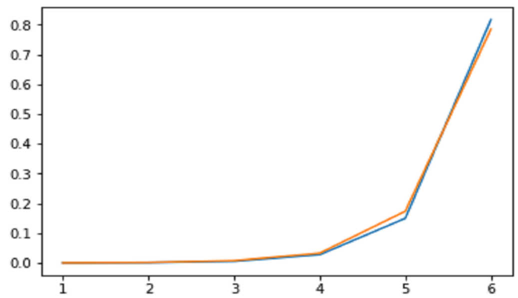
Last but not least, the parameters u and λ_t of the limit multi-alternative choice distribution appearing in Theorem 4 are those that govern the pairwise comparisons that lead to it, and are thus identified by Theorems 1 and 3.

[25] As usual, p_C is the conditional of p_A given C. That is, $p_C\left(a\right) = e^{\lambda_t u(a)} / \sum_{c \in C} e^{\lambda_t u(c)}$ if $a \in C$, and $p_C\left(a\right) = 0$ else.

[26] This is the case considered by Cerreia-Vioglio et al. [6], who also restrict their attention to the DDM only.

Simulations and final considerations Going back one last time to our snacks' example, recall that an agent with a deadline t of 4 seconds selected a consideration set C of 6 sweet snacks:

At this point, we initialize the Metropolis-DDM algorithm with the (binary) EZ-DDM parameters experimentally obtained by Milosavljevic et al. [34],[27] and run it:

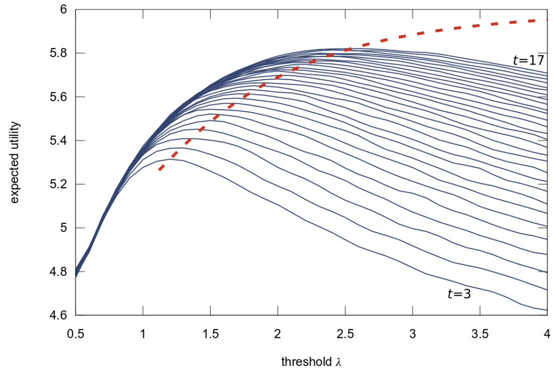


In the plot, on the x-axis the utilities $1, 2, ..., 6$ of the 6 alternatives are listed, and the y-axis reports the alternative's choice frequencies (with $\lambda = 1.2$). The theoretical softmax distribution is plotted in blue, the output of the Metropolis-DDM algorithm in orange. Numerical convergence of the simulated choice distribution to softmax is evident.

Although our agent initially ignores the true value of the alternatives and "discovers" it through DDM comparison, in circa the 80% of the cases they behave like the neo-classical utility maximizer (who always chooses the best alternative in virtual time). The natural question regards then the optimality of the procedure we propose: What is the optimal error rate for this algorithm? Very low error rates – that is, very large values of λ – prevent the algorithm from exploring the whole consideration set before deadline t is reached (because pairwise comparisons take too much time), while very high error rates – that is, very small values of λ – amount to almost uniformly random choice. Thus, an efficient adjustment of the parameter λ must solve this speed-accuracy tradeoff.

Numerical simulations suggest that, for each t, there exists a unique optimal $\lambda^* = \lambda^*(t)$ and that $\lambda^*(t)$ is an increasing and concave function of t – depicted as a red dashed curve below. In the picture, the expected utility produced by the algorithm is plotted as a function of λ for $t = 3, 4, ..., 17$. Notice that an expected utility of more than 5.33 (resp., 5.35) is achieved in 3 (resp., 4) seconds with a λ^* of approximately 1.2 (resp., 1.3), etc.

[27] The estimates of Milosavljevic et al. [34] correspond to utilities that range between 0 and 7.071 and λs that range between 0.849 to 1.442 for speeded binary comparisons. Here we choose utilities $1, 2, ..., 6$ for our alternatives and $\lambda = 1.2$. Codes and simulations are available at `https://github.com/carlobaldassi/MetropolisDDM_python` (based on Drugowitsch [10]).



So, the simulation shows that the performance of the Metropolis-DDM algorithm, measured by the distance between its expected payoff and $\max_{a \in A} u(a)$, is very good, even under severe time pressure. To the best of our knowledge, formal optimality results for the classical multi-alternative version of the DDM (called MDDM) are not available, while asymptotic results that apply to negligible error rates are not particularly useful for observed behavior (see, e.g., McMillen and Holmes [33] and Ditterich [9]). At the same time, as we already discussed, working memory can maintain representations of only 3 to 4 objects at any given moment. This makes the MDDM an implausible process of multi-alternative choice for menus of more than 4 items – as discussed in Krajbich and Rangel [23, p. 13856].[28] The Metropolis-DDM algorithm, instead, needs only the representation of incumbent and proposal in working memory, and compares them in the fastest possible way, for given error rate.

5 Acknowledgements

The authors thank Pierpaolo Battigalli, Manel Baucells, Rafal Bogacz, Andrei Borodin, Giacomo Cattelan, Roberto Corrao, Jochen Ditterich, Loic Grenie, Philip Holmes, Giacomo Lanzani, Paolo Leonetti, Antonio Rangel, Aldo Rustichini, Michael Shadlen for very helpful suggestions, and especially Marco Pirazzini for brilliant research assistance and for his crucial role in simulations and coding.

Simone Cerreia-Vioglio, and Fabio Maccheroni and Massimo Marinacci gratefully acknowledge the financial support of ERC (grants SDDM-TEA and INDIMACRO, respectively).

6 Appendix: Proofs

For brevity, sometimes we write $\overline{X}$ instead of $\mathbb{E}[X]$, for example we use $\overline{\mathrm{DT}}$ and MDT interchangeably.

Proof of Theorem 1 (i) implies (ii). By positivity and the definition of observables, we have that $R^o_{a,b} > 0$, for all $a \neq b$ in A. Arbitrarily choose $c \in A$, set $v(c) \equiv 0$ and

$$v(a) \equiv \log R^o_{a,c} \tag{5}$$

for all $a \neq c$ in A. Then, for all $a \neq b$ in $A \setminus \{c\}$, by the product rule, we have

$$R^o_{a,b} = R^o_{a,c} R^o_{c,b} = \frac{R^o_{a,c}}{R^o_{b,c}} = e^{v(a)-v(b)}$$

[28] See also Cerreia-Vioglio et al. [6].

and direct application of (5) delivers the same result for $a = c \neq b$ and for $b = c \neq a$. Then $P^o_{a,b} = 1/\left(1 + \left(R^o_{a,b}\right)^{-1}\right)$ implies

$$P^o_{a,b} = \frac{1}{1 + e^{-[v(a)-v(b)]}}$$

for all $a \neq b$ in A.

Tedious verification shows that invariance guarantees that

$$\overline{\mathrm{DT}}^o_{a,b} \frac{\operatorname{logit}\left(1 - \mathrm{ER}^o_{a,b}\right)}{1/2 - \mathrm{ER}^o_{a,b}} = \overline{\mathrm{DT}}^o_{a',b'} \frac{\operatorname{logit}\left(1 - \mathrm{ER}^o_{a',b'}\right)}{1/2 - \mathrm{ER}^o_{a',b'}} \tag{6}$$

for all $a \neq b$ and all $a' \neq b'$ in A. [29]

Now arbitrarily choosing $a' \neq b'$ in A, and setting

$$\lambda^2 \equiv \overline{\mathrm{DT}}^o_{a',b'} \frac{\operatorname{logit}\left(1 - \mathrm{ER}^o_{a',b'}\right)}{1 - 2\mathrm{ER}^o_{a',b'}} = \overline{\mathrm{DT}}^o_{a',b'} \frac{\log\left(\frac{1 - \frac{1}{1+e^{|v(a')-v(b')|}}}{\frac{1}{1+e^{|v(a')-v(b')|}}}\right)}{1 - \frac{2}{1+e^{|v(a')-v(b')|}}} = \overline{\mathrm{DT}}^o_{a',b'} \frac{e^{|v(a')-v(b')|} + 1}{e^{|v(a')-v(b')|} - 1} |v(a') - v(b')| \tag{7}$$

and $u(a) \equiv v(a)/\lambda$ for all a in A, it follows that, for all $a \neq b$ in A,

$$P^o_{a,b} = \frac{1}{1 + e^{-[v(a)-v(b)]}} = \frac{1}{1 + e^{-\lambda[u(a)-u(b)]}} = P_{a,b}$$

and

$$\begin{aligned}
\overline{\mathrm{DT}}^o_{a,b} &= \lambda^2 \frac{e^{|v(a)-v(b)|} - 1}{e^{|v(a)-v(b)|} + 1} \frac{1}{|v(a) - v(b)|} = \lambda^2 \frac{e^{\lambda|u(a)-u(b)|} - 1}{e^{\lambda|u(a)-u(b)|} + 1} \frac{1}{\lambda |u(a) - u(b)|} \\
&= \frac{\lambda}{|u(a) - u(b)|} \frac{e^{\lambda|u(a)-u(b)|} - 1}{e^{\lambda|u(a)-u(b)|} + 1} = \frac{\lambda}{|u(a) - u(b)|} \tanh\left(\frac{\lambda |u(a) - u(b)|}{2}\right) \\
&= \frac{\lambda}{u(a) - u(b)} \tanh\left(\frac{\lambda [u(a) - u(b)]}{2}\right) = \overline{\mathrm{DT}}_{a,b}
\end{aligned}$$

where the first equality is a consequence of (6) and (7). Thus (ii) holds for the DDM with parameters u, $\sigma = 1$, and λ.

Verifying that (ii) implies (i) is simple and so omitted for brevity.

Finally, if (ii) holds, by (1), we have that, for all $a \neq b$ in A,

$$\begin{aligned}
\operatorname{logit} P^o_{a,b} &= \operatorname{logit} P_{a,b} = \log \frac{P_{a,b}}{P_{b,a}} = \log \frac{\frac{1}{1 + e^{-\frac{\lambda}{\sigma^2}[u(a)-u(b)]}}}{\frac{1}{1 + e^{-\frac{\lambda}{\sigma^2}[u(b)-u(a)]}}} \\
&= \frac{\lambda}{\sigma^2} [u(a) - u(b)] = \frac{\lambda}{\sigma} \left[\frac{u}{\sigma}(a) - \frac{u}{\sigma}(b)\right]
\end{aligned}$$

and the second part of (3) follows; the first part is a consequence of

$$\begin{aligned}
\overline{\mathrm{DT}}^o_{a,b} &= \overline{\mathrm{DT}}_{a,b} = \frac{\lambda}{u(a) - u(b)} \tanh\left(\frac{\lambda [u(a) - u(b)]}{2\sigma^2}\right) = \frac{\lambda}{|u(a) - u(b)|} \tanh\left(\frac{\lambda |u(a) - u(b)|}{2\sigma^2}\right) \\
&= \frac{\hat{\lambda}}{|\hat{u}(a) - \hat{u}(b)|} \tanh\left(\frac{\hat{\lambda} |\hat{u}(a) - \hat{u}(b)|}{2}\right) = \hat{\lambda}^2 \frac{1 - 2\frac{1}{1+\exp\left(\hat{\lambda}|\hat{u}(a)-\hat{u}(b)|\right)}}{\hat{\lambda} |\hat{u}(a) - \hat{u}(b)|} = \hat{\lambda}^2 \frac{2\left(\frac{1}{2} - \mathrm{ER}^o_{a,b}\right)}{\operatorname{logit}\left(1 - \mathrm{ER}^o_{a,b}\right)}
\end{aligned}$$

[29] And not only if $a' = a$ as the axiom requires.

As wanted. $\blacksquare$

Proof of Proposition 2 Arbitrarily choose $u \in \mathbb{R}^A$, $\lambda > 0$ ($\sigma = 1$), and $a, b \in A$. Let $\Delta = u(a) - u(b)$. Since we will repeatedly use the results of the Handbook of Brownian Motion of Borodin and Salminen [4], henceforth HBM, we adopt their notation. Specifically, setting $\mu = \Delta/\sqrt{2}$ and $z = \lambda/\sqrt{2}$,

$$\frac{Z_{a,b}(s)}{\sqrt{2}} = \mu s + W(s) \overset{\text{HBM}}{=} W_s^{(\mu)}$$

$$\mathrm{DT}_{a,b} = \min\{s : |Z_{a,b}(s)| = \lambda\} = \min\left\{s : \left|W_s^{(\mu)}\right| = z\right\} \overset{\text{HBM}}{=} H_{-z,z}$$

$$\mathrm{DO}_{a,b} = \begin{cases} a & \text{if } \dfrac{Z_{a,b}(\mathrm{DT}_{a,b})}{\sqrt{2}} = \dfrac{\lambda}{\sqrt{2}} \\ b & \text{if } \dfrac{Z_{a,b}(\mathrm{DT}_{a,b})}{\sqrt{2}} = -\dfrac{\lambda}{\sqrt{2}} \end{cases} = \begin{cases} a & \text{if } W_{H_{-z,z}}^{(\mu)} = z \\ b & \text{if } W_{H_{-z,z}}^{(\mu)} = -z \end{cases}$$

With this, their Equation 3.0.2 (p. 233) shows that

$$\mathbb{P}[H_{-z,z} \in dt] = e^{-\frac{\mu^2 t}{2}} \left(e^{-\mu z} + e^{\mu z}\right) \mathrm{ss}_{z,2z}(t)\, dt \tag{8}$$

where $\mathrm{ss}_{z,2z}(t)$ is defined on p. 451 of HBM. Their Equation 3.0.4(b) (p. 233) yields

$$\mathbb{P}\left[W_{H_{-z,z}}^{(\mu)} = z\right] = \frac{e^{\mu z}}{e^{-\mu z} + e^{\mu z}}$$

while Equation 3.0.6(b) (p. 233) gives

$$\mathbb{P}\left[H_{-z,z} \in dt, W_{H_{-z,z}}^{(\mu)} = z\right] = e^{\mu z} e^{-\frac{\mu^2 t}{2}} \mathrm{ss}_{z,2z}(t)\, dt = \mathbb{P}[H_{-z,z} \in dt]\, \mathbb{P}\left[W_{H_{-z,z}}^{(\mu)} = z\right]$$

This proves that $\mathrm{DT}_{a,b}$ and $\mathrm{DO}_{a,b}$ are independent random variables, because $\mathrm{DT}_{a,b} = H_{-z,z}$ and $\mathrm{DO}_{a,b}$ only depends on whether $W_{H_{-z,z}}^{(\mu)} = z$ or $W_{H_{-z,z}}^{(\mu)} = -z$.

As to the equivalence between (i)-(iv), by (8), the density of $\mathrm{DT}_{a,b}$ is

$$f_{\mathrm{DT}_{a,b}}(t) = \frac{\lambda e^{-\frac{\Delta^2 t}{4}}}{\sqrt{\pi} t^{3/2}} \cosh\left(\frac{\lambda \Delta}{2}\right) \sum_{k=-\infty}^{\infty} (1 + 4k)\, e^{-\frac{\lambda^2}{4t}(1+4k)^2} \qquad \forall t \in (0, \infty) \tag{9}$$

but, for all $q \in (0,1)$, $\sum_{k=-\infty}^{\infty} (1+4k)\, q^{\frac{1}{4}(1+4k)^2} = \sqrt[4]{q} \sum_{n=0}^{\infty} (-)^n (2n+1)\, q^{n(n+1)} = \vartheta_1'(0,q)/2$ where ϑ_1 is the first Jacobi theta function. Thus setting $y = |\Delta|$, we have

$$f_{\mathrm{DT}_{a,b}}(t) = f(t,y) = e^{-\frac{y^2 t}{4}} \cosh\left(\frac{\lambda y}{2}\right) \frac{1}{2\sqrt{\pi}} \frac{\lambda}{t^{3/2}} \vartheta_1'\left(0, e^{-\frac{\lambda^2}{t}}\right) \qquad \forall t \in (0, \infty)$$

which is continuous and bounded, as a function of (t,y), on every rectangle $T_x \times Y = (0,x) \times [0, \max_A u - \min_A u]$ with $x \in (0, \infty)$.

Now, for each (fixed) $x \in (0, \infty)$, the distribution function of $\mathrm{DT}_{a,b}$ is

$$F_{\mathrm{DT}_{a,b}}(x) = F(x,y) = \int_0^x f(t,y)\, dt$$

and it is continuous on Y because $f(t,y)$ is continuous and bounded on $T_x \times Y$. Moreover,

$$\frac{\partial f}{\partial y}(t,y) = \frac{y}{2}\left(\frac{\lambda}{y} \tanh\left(\frac{\lambda y}{2}\right) - t\right) f(t,y) \qquad \forall (t,y) \in T_x \times \mathrm{int}(Y)$$

is continuous and bounded too.

Differentiation under the integral sign is then possible, and it shows that, for all $y \in \operatorname{int}(Y)$,

$$\frac{\partial F}{\partial y}(x,y) = \int_0^x \frac{\partial f}{\partial y}(t,y)\,dt = \frac{y}{2}\int_0^x \left(\overline{\mathrm{DT}}_{a,b} - t\right) f_{\mathrm{DT}_{a,b}}(t)\,dt$$

For $x < \overline{\mathrm{DT}}_{a,b}$ the integrand is positive, and so is $\partial F/\partial y$. While, for $x \geq \overline{\mathrm{DT}}_{a,b}$

$$\begin{aligned}\int_0^x \left(\overline{\mathrm{DT}}_{a,b} - t\right) f_{\mathrm{DT}_{a,b}}(t)\,dt &= \int_0^{\overline{\mathrm{DT}}_{a,b}} \left(\overline{\mathrm{DT}}_{a,b} - t\right) f_{\mathrm{DT}_{a,b}}(t)\,dt + \int_{\overline{\mathrm{DT}}_{a,b}}^x \left(\overline{\mathrm{DT}}_{a,b} - t\right) f_{\mathrm{DT}_{a,b}}(t)\,dt \\ &\geq \int_0^{\overline{\mathrm{DT}}_{a,b}} \left(\overline{\mathrm{DT}}_{a,b} - t\right) f_{\mathrm{DT}_{a,b}}(t)\,dt + \int_{\overline{\mathrm{DT}}_{a,b}}^\infty \left(\overline{\mathrm{DT}}_{a,b} - t\right) f_{\mathrm{DT}_{a,b}}(t)\,dt = 0\end{aligned}$$

where, in the second line, inequality holds because the integrand of the second summand is negative and the final equality holds because $\int_0^\infty \left(\overline{\mathrm{DT}}_{a,b} - t\right) f_{\mathrm{DT}_{a,b}}(t)\,dt = \mathbb{E}\left[\overline{\mathrm{DT}}_{a,b} - \mathrm{DT}_{a,b}\right]$, and again $\partial F/\partial y$ is positive. Summing up, for each (fixed) $x \in (0,\infty)$, $F(x,y)$ is continuous on $[0, \max_A u - \min_A u]$ and differentiable on $(0, \max_A u - \min_A u)$ with respect to y, and positivity of the derivative yields monotonicity (for fixed x, with respect to $y = |u(a) - u(b)|$).

But this shows that if $|u(a) - u(b)| \leq |u(a') - u(b')|$, $F_{\mathrm{DT}_{a,b}}(x) \leq F_{\mathrm{DT}_{a',b'}}(x)$ for all $x \in (0,\infty)$, that is, $\mathrm{DT}_{a,b}$ stochastically dominates $\mathrm{DT}_{a',b'}$.

Then (i) implies (iv). On the other hand, if $\mathrm{DT}_{a,b}$ stochastically dominates $\mathrm{DT}_{a',b'}$, then obviously $\overline{\mathrm{DT}}_{a,b} \geq \overline{\mathrm{DT}}_{a',b'}$, so that (iv) implies (iii). Moreover, $\overline{\mathrm{DT}}_{a,b} \geq \overline{\mathrm{DT}}_{a',b'}$ implies

$$\frac{\lambda}{|\Delta|}\tanh\left(\frac{\lambda|\Delta|}{2}\right) = \frac{\lambda}{\Delta}\tanh\left(\frac{\lambda\Delta}{2}\right) \geq \frac{\lambda}{\Delta'}\tanh\left(\frac{\lambda\Delta'}{2}\right) = \frac{\lambda}{|\Delta'|}\tanh\left(\frac{\lambda|\Delta'|}{2}\right)$$

whence $|\Delta| \leq |\Delta'|$ because $(\lambda/y)\tanh(\lambda y/2)$ is strictly decreasing, for fixed $\lambda > 0$, and $y \in [0,\infty)$; but – in turn – this implies

$$\mathrm{ER}_{a,b} = \frac{1}{1+e^{\lambda|\Delta|}} \geq \frac{1}{1+e^{\lambda|\Delta'|}} = \mathrm{ER}_{a',b'}$$

and (iii) implies (ii). Finally, (ii) implies (i) because

$$\frac{1}{1+e^{\lambda|\Delta|}} = \mathrm{ER}_{a,b} \geq \mathrm{ER}_{a',b'} = \frac{1}{1+e^{\lambda|\Delta'|}} \implies |\Delta| \leq |\Delta'|$$

As wanted. ■

Proof of Theorem 3 (i) implies (ii). By EZ1, we have that $R^o_{a,b} > 0$, for all $a \neq b$ in A. Arbitrarily choose $c \in A$, set $v(c) \equiv 0$ and

$$v(a) \equiv \log R^o_{a,c} \tag{10}$$

for all $a \neq c$ in A. Then, for all $a \neq b$ in $A \setminus \{c\}$, by EZ2, we have

$$R^o_{a,b} = R^o_{a,c}R^o_{c,b} = \frac{R^o_{a,c}}{R^o_{b,c}} = e^{v(a)-v(b)}$$

and direct application of (10) delivers the same result for $a = c \neq b$ and for $b = c \neq a$. Then $P^o_{a,b} = 1/\left(1 + \left(R^o_{a,b}\right)^{-1}\right)$ implies

$$P^o_{a,b} = \frac{1}{1+e^{-[v(a)-v(b)]}}$$

for all $a \neq b$ in A.

Tedious verification shows that axiom EZ3 guarantees that

$$\mathrm{VRT}^o_{a,b} \frac{\left(R^o_{a,b}+1\right)^2 \left(\log R^o_{a,b}\right)^3}{\left(R^o_{a,b}\right)^2 - 2R^o_{a,b}\log R^o_{a,b} - 1} = \mathrm{VRT}^o_{a',b'} \frac{\left(R^o_{a',b'}+1\right)^2 \left(\log R^o_{a',b'}\right)^3}{\left(R^o_{a',b'}\right)^2 - 2R^o_{a',b'}\log R^o_{a',b'} - 1} \tag{11}$$

for all $a \neq b$ and all $a' \neq b'$ in A. [30]

Now arbitrarily choosing $a' \neq b'$ in A, and setting

$$2\lambda^4 \equiv \mathrm{VRT}^o_{a',b'} \frac{\left(R^o_{a',b'}+1\right)^2 \left(\log R^o_{a',b'}\right)^3}{\left(R^o_{a',b'}\right)^2 - 2R^o_{a',b'}\log R^o_{a',b'} - 1} \tag{12}$$

and $u(a) \equiv v(a)/\lambda$ for all a in A, it follows that, for all $a \neq b$ in A,

$$P^o_{a,b} = \frac{1}{1+e^{-[v(a)-v(b)]}} = \frac{1}{1+e^{-\lambda[u(a)-u(b)]}} = P_{a,b}$$

and

$$\mathrm{VRT}^o_{a,b} = 2\lambda^4 \frac{\left(R^o_{a,b}\right)^2 - 2R^o_{a,b}\log R^o_{a,b} - 1}{\left(R^o_{a,b}+1\right)^2 \left(\log R^o_{a,b}\right)^3} = 2\lambda^4 \frac{\left(R_{a,b}\right)^2 - 2R_{a,b}\log R_{a,b} - 1}{\left(R_{a,b}+1\right)^2 \left(\log R_{a,b}\right)^3} = \mathrm{VRT}_{a,b} \tag{13}$$

where $P_{a,b}$ and $\mathrm{VRT}_{a,b}$ are the theoretical choice frequency and the theoretical variance of $\mathrm{RT}_{a,b}$ of any $\mathrm{DM}(u,\lambda,*)$ with u and λ chosen as above.[31] By axiom EZ4, the quantity

$$T_{a,b} \equiv \mathrm{MRT}^o_{a,b} - \sqrt{\mathrm{VRT}^o_{a,b} \frac{\left(R^o_{a,b}-1\right)^2 \log R^o_{a,b}}{2\left(R^o_{a,b}\right)^2 - 4R^o_{a,b}\log R^o_{a,b} - 2}}$$

is positive for all $a \neq b$ in A,[32] and – when $\mathrm{DM}(u,\lambda,T)$ is considered – we have

$$\begin{aligned}
\mathrm{MRT}_{a,b} &= T_{a,b} + \mathrm{MDT}_{a,b} \\
&= \mathrm{MRT}^o_{a,b} - \sqrt{\mathrm{VRT}^o_{a,b} \frac{\left(R^o_{a,b}-1\right)^2 \log R^o_{a,b}}{2\left(R^o_{a,b}\right)^2 - 4R^o_{a,b}\log R^o_{a,b} - 2}} + \frac{\lambda}{u(a)-u(b)} \tanh\left(\frac{\lambda\left[u(a)-u(b)\right]}{2}\right) \\
&= \mathrm{MRT}^o_{a,b} - \sqrt{\mathrm{VRT}^o_{a,b} \frac{\left(R^o_{a,b}-1\right)^2 \log R^o_{a,b}}{2\left(R^o_{a,b}\right)^2 - 4R^o_{a,b}\log R^o_{a,b} - 2}} + \sqrt{\lambda^4 \frac{\left(R_{a,b}-1\right)^2}{\left(R_{a,b}+1\right)^2} \frac{1}{\left(\log R_{a,b}\right)^2}} \\
&= \mathrm{MRT}^o_{a,b} - \sqrt{\mathrm{VRT}^o_{a,b} \frac{\left(R^o_{a,b}-1\right)^2 \log R^o_{a,b}}{2\left(R^o_{a,b}\right)^2 - 4R^o_{a,b}\log R^o_{a,b} - 2}} + \sqrt{\lambda^4 \frac{\left(R^o_{a,b}-1\right)^2}{\left(R^o_{a,b}+1\right)^2} \frac{1}{\left(\log R^o_{a,b}\right)^2}}
\end{aligned}$$

[30] And not only if $a' = a$ as the axiom requires.

[31] The first equality is a consequence of (11)-(12) and the last one follows from the results of [55].

[32] Also notice that, $T_{a,b} = T_{b,a}$. Moreover $T_{a,b}$ is independent of the pair a, b if and only if the assumption:

$$\mathrm{MRT}^o_{a,b} - \sqrt{\mathrm{VRT}^o_{a,b} \frac{\left(R^o_{a,b}-1\right)^2 \log R^o_{a,b}}{2\left(R^o_{a,b}\right)^2 - 4R^o_{a,b}\log R^o_{a,b} - 2}} = \mathrm{MRT}^o_{a,c} - \sqrt{\mathrm{VRT}^o_{a,c} \frac{\left(R^o_{a,c}-1\right)^2 \log R^o_{a,c}}{2\left(R^o_{a,c}\right)^2 - 4R^o_{a,c}\log R^o_{a,c} - 2}}$$

for all distinct $a, b, c \in A$, is added to EZ1–EZ4.

but, by (13), it follows

$$\begin{aligned}
\lambda^4 \frac{\left(R^o_{a,b}-1\right)^2}{\left(R^o_{a,b}+1\right)^2} \frac{1}{\left(\log R^o_{a,b}\right)^2} &= \frac{\mathrm{VRT}^o_{a,b}}{2} \frac{\left(R^o_{a,b}+1\right)^2 \left(\log R^o_{a,b}\right)^3}{\left(R^o_{a,b}\right)^2 - 2R^o_{a,b}\log R^o_{a,b} - 1} \frac{\left(R^o_{a,b}-1\right)^2}{\left(R^o_{a,b}+1\right)^2} \frac{1}{\left(\log R^o_{a,b}\right)^2} \\
&= \mathrm{VRT}^o_{a,b} \frac{\left(R^o_{a,b}-1\right)^2 \log R^o_{a,b}}{2\left(R^o_{a,b}\right)^2 - 4R^o_{a,b}\log R^o_{a,b} - 2}
\end{aligned}$$

and so $\mathrm{MRT}_{a,b} = \mathrm{MRT}^o_{a,b}$. Thus (ii) holds for the EZ-DDM with parameters u, $\sigma = 1$, λ, and T. Verifying that (ii) implies (i) and checking the uniqueness properties of the parameters is a long exercise, based on our previous analysis and the results of [55]. ■

Proof of Theorem 4 In the proof we assume

$$P_{a,b} = \begin{cases} \dfrac{1 - e^{-\beta[u(a)-u(b)]}}{1 - e^{-(\lambda+\beta)[u(a)-u(b)]}} & \text{if } u(a) \neq u(b) \\ \dfrac{\beta}{\lambda+\beta} & \text{if } u(a) = u(b) \end{cases} \in (0,1) \tag{14}$$

thus allowing for asymmetric lower and upper barriers,[33] $-\beta < 0$ and $\lambda > 0$, respectively.

The explicit form of $M = M_t$ (the subscript t will be omitted throughout) is

$$M_{ba} = M(a \mid b) = \begin{cases} Q(a \mid b) P_{a,b} & \text{if } a \neq b \\ 1 - \sum_{c \in A \setminus \{b\}} Q(c \mid b) P_{c,b} & \text{if } a = b \end{cases} \tag{15}$$

and this allows to show that M is a *bona fide* stochastic matrix.

Next we show that M is reversible with respect to $p_A = p_A^{(u,\lambda)}$. Let $a \neq b$ in A.

- If $u(a) - u(b) \neq 0$, then

$$\begin{aligned}
M(a \mid b) p_A(b) &= \frac{Q(a \mid b)}{\sum_{x \in A} e^{\lambda u(x)}} \cdot \frac{e^{\lambda u(b)} - e^{-\beta u(a) + \beta u(b) + \lambda u(b)}}{1 - e^{-(\lambda+\beta)[u(a)-u(b)]}} \\
&= \frac{Q(b \mid a)}{\sum_{x \in A} e^{\lambda u(x)}} \cdot \frac{e^{\lambda u(a)} - e^{-\beta u(b) + \beta u(a) + \lambda u(a)}}{1 - e^{-(\lambda+\beta)[u(b)-u(a)]}} = M(b \mid a) p_A(a)
\end{aligned}$$

  because Q is symmetric and

$$\frac{e^{\lambda u(b)} - e^{-\beta u(a) + \beta u(b) + \lambda u(b)}}{1 - e^{-(\lambda+\beta)[u(a)-u(b)]}} = \frac{e^{\lambda u(a)} - e^{-\beta u(b) + \beta u(a) + \lambda u(a)}}{1 - e^{-(\lambda+\beta)[u(b)-u(a)]}}$$

- Else $u(a) - u(b) = 0$, that is, $u(a) = u(b)$, then

$$\begin{aligned}
M(a \mid b) p_A(b) &= Q(a \mid b) \frac{\beta}{\lambda+\beta} \frac{e^{\lambda u(b)}}{\sum_{x \in A} e^{\lambda u(x)}} \\
&= Q(b \mid a) \frac{\beta}{\lambda+\beta} \frac{e^{\lambda u(a)}}{\sum_{x \in A} e^{\lambda u(x)}} = M(b \mid a) p_A(a)
\end{aligned}$$

  because Q is symmetric.

[33] See footnote 7.

Since $M(a \mid b) p_A(b) = M(b \mid a) p_A(a)$ also if $a = b$, then reversibility holds.

It is then easy to see that, if Q is irreducible, then M is irreducible and aperiodic. In turn this implies that p_A is its stationary distribution and therefore $\mu M^n \to p_A$ as $n \to \infty$ for all $\mu \in \Delta(A)$ (see Madras [30, Ch. 4]).[34]

If instead Q is reducible, since it is symmetric then all communicating classes are closed (see Norris [36, Ch. 1]). In fact, if $Q_{a_1 a_2} Q_{a_2 a_3} \dots Q_{a_{m-1} a_m} > 0$, then $Q_{a_m a_{m-1}} Q_{a_{m-1} a_{m-2}} \dots Q_{a_2 a_1} > 0$ and $a_1 \to a_m$ implies $a_m \to a_1$. Rearrange the alternatives so that the communicating classes are

$$A_1 = \{1, ..., |A_1|\}, A_2 = \{|A_1| + 1, ..., |A_1| + |A_2|\}, \dots, A_K = \{|A| - |A_K| + 1, ..., |A|\}$$

Notice that given any class A_k and any $b \in A_k$, then $Q(a \mid b) = Q_{ba} = 0$ for all $a \notin A_k$,[35] thus for all the rows belonging to A_k the only nonzero elements are in columns belonging to A_k (and also the converse is true by symmetry). That is, $Q = \operatorname{diag}(Q_1, \dots, Q_K)$ is a block diagonal matrix; moreover, by definition of communicating classes all the Q_k are irreducible (stochastic and symmetric). Now by (15) also $M = \operatorname{diag}(M_1, \dots, M_K)$ is block diagonal. By the first part of this proof, each of the M_k's is aperiodic, irreducible, with stationary distribution given by the restriction p_k of p_{A_k} to A_k. Then (see again Madras [30, Th. 4.2])

$$M_k^n \to \begin{bmatrix} p_k \\ p_k \\ \vdots \\ p_k \end{bmatrix} \equiv \Pi_k \qquad \forall k = 1, ..., K$$

now let $\mu = \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_K \end{bmatrix} \in \Delta(A)$ with $\mu_k \in \mathbb{R}_+^{|A_k|}$ for all $k = 1, ..., K$. Given any $k = 1, ..., K$, since $\mu_k \Pi_k$ is the linear combination of the lines of Π_k with weights given by μ_k,

$$\mu_k M_k^n \to \mu_k \Pi_k = \mu_{k1} p_k + \mu_{k2} p_k + \dots + \mu_{k|A_k|} p_k = \mu(A_k) p_k$$

therefore, by block-multiplication,

$$\mu M^n = \begin{bmatrix} \mu_1 M_1^n & \mu_2 M_2^n & \cdots & \mu_K M_K^n \end{bmatrix} \to \begin{bmatrix} \mu(A_1) p_1 & \mu(A_2) p_2 & \cdots & \mu(A_K) p_K \end{bmatrix}$$

and so $\mu M^n(a) \to \sum_{k=1}^{K} \mu(A_k) p_{A_k}(a)$ for all a in A. As wanted. ■

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[34] As usual, $\Delta(A)$ is the set of all probability distributions on A.

[35] Otherwise, we would have $b \in A_k$ and $b \to a$, which by closure would imply $a \in A_k$, a contradiction.

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